EXTREMAL NEGATIVITY IN SYMMETRIC-GROUP CHARACTER TABLES: BINARY COMPRESSION AND TERMINAL HOOK POSITIVITY

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Abstract. Let $N_n^-(\lambda)$ denote the number of conjugacy classes of the symmetric group S_n on which the irreducible character χ^λ is negative, with classes counted without size weights. Hopkins' problem OPAC-038 asks whether, for all sufficiently large n, this statistic is maximized by the sign representation. The global problem remains open.

We develop a direct binary-compression route for the balanced hook $\lambda_k = (k+1, 1^{k-1}) \vdash 2k$. The hook generating function and the Glaisher split–merge graph reduce the parity-balance deficit to a count of canonical binary edges whose two endpoint values are simultaneously negative versus simultaneously nonnegative. For a canonical edge

$$A = \tau \cup (m, m), \qquad B = \tau \cup (2m),$$

with $q = k - m$ and $d = k - 2m$, we obtain the exact two-scale formula

$$\chi(A) = \varepsilon(\tau)H_d(\tau) - 2(-1)^m H_{q-1}(\tau) + H_{d-1}(\tau), \qquad \chi(B) = \varepsilon(\tau)H_d(\tau) - H_{d-1}(\tau).$$

When the terminal scale m is odd, every strict negative–negative edge with $m > d$ has exactly one higher even part $2b$, and the same-scale split $2b \mapsto b + b$ sends its endpoint pair from $(-D, -P)$ to $(P + 4C, D)$ with $C \geq 0$. Hence every terminal odd scale $m > k/3$ is paid injectively, in all degrees and without an asymptotic argument.

We then treat the first open band $k/4 < m \leq k/3$. Every strict negative–negative edge is proved to have exactly one even part $2b$, except for the isolated family $\tau = (2b, 2b)$, which has value pair $(-2, -2)$ and is paid explicitly by $\tau^+ = (2b, b, b)$ with value pair $(0, 0)$. For every remaining one-even source $\tau = \sigma \cup (2b)$ we derive an exact finite-difference normal form and construct an explicit family of all-odd zero targets

$$Z_R = R \cup (2q - \mathrm{wt}(R)),$$

for odd-cardinality submultisets $R \subseteq \sigma$ of weight below d. Thus every source in the first transition band has at least one favorable target; the unresolved issue is only collision control. We exhibit exact counterexamples showing that odd splits alone and zero absorption alone are each insufficient, and formulate the resulting two-tier split-or-absorb Hall problem. Exact integer computation verifies that hybrid architecture through odd $m \leq 23$ on 14,837 one-even negative–negative sources. We also record an exact mixed binary-square identity that identifies the natural aggregate mechanism for the still-open even 2-adic scales.

1. Introduction

Let $\lambda, \mu \vdash n$, and write $\chi^\lambda(\mu)$ for the ordinary irreducible character value of S_n. We study the unweighted sign statistic

$$N_n^-(\lambda) = \#\{\mu \vdash n : \chi^\lambda(\mu) < 0\}.$$

The sign representation has no zero entries and is negative precisely on the odd conjugacy classes. Hopkins' OPAC-038 asks whether

$$N_n^-(\lambda) \leq N_n^-((1^n)) \tag{1.1}$$

for every $\lambda \vdash n$ once n is sufficiently large [1]. Exact computation currently verifies (1.1) through $n = 29$, but no all-degree or eventual proof is known.

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The purpose of this paper is not to assemble every structural route to OPAC-038. Instead, we isolate the cleanest direct mechanism currently available in the hook sector and push it to its precise collision frontier. The central object is the balanced hook

$$\lambda_k = (k+1, 1^{k-1}) \vdash 2k.$$

For this row, binary cycle merging interacts particularly well with the hook-character generating function. The resulting Glaisher graph converts the sign problem into a local comparison on split–merge edges, but crucially it does so without replacing the original unweighted count by a magnitude-weighted statistic.

The main proved results are as follows.

(i) We derive an exact two-scale formula for every canonical binary edge of the balanced hook.
(ii) Every terminal odd scale $m > k/3$ contributes nonpositively to the balanced parity deficit. More strongly, every negative–negative edge at such a scale is sent injectively to a strict positive–positive edge by one further Glaisher split.
(iii) In the first transition band $k/4 < m \leq k/3$, every strict negative–negative edge contains exactly one even part, except for a single explicit double-even family. The exceptional family is paid exactly.
(iv) Every remaining one-even source in that band possesses an explicit all-odd $(0,0)$ target. Hence local target existence is completely solved in the first transition band; only global distinctness remains.
(v) The two most natural stronger matching statements are false. We give a source for which no odd split is favorable, and a separate band in which the complete zero-absorption target set has smaller cardinality than the source set.
(vi) The surviving frontier is a two-tier Hall problem with disjoint split and absorption target spaces. Exact arithmetic verifies the proposed architecture through odd $m \leq 23$.

These results sit inside a larger program. Previous work established the exact parity-balance reformulation, every self-conjugate row, a uniform two-row boundary strip and the first three two-row boundary pairs, the complete second exterior-hook conjugate pair, several zero-reservoir and square-core theorems, and a final-defect barrier of order $n^{2/3}$; see [12]. Those results are recalled briefly in Section 11, but the proofs below are organized so that the new hook argument can be read independently.

2. Parity balance and the exact endpoint

For a partition $\mu \vdash n$, put

$$\varepsilon(\mu) = (-1)^{n-\ell(\mu)}.$$

Let $\mathcal{E}_n$ and $\mathcal{O}_n$ be the sets of even and odd conjugacy classes. Let $p(n)$ be the partition number and $q(n)$ the number of self-conjugate partitions, equivalently the number of partitions of n into distinct odd parts.

Proposition 2.1 (Sign-row count)**.** *For every $n \geq 0$,*

$$\sum_{\mu \vdash n} \varepsilon(\mu) = q(n), \qquad N_n^-((1^n)) = \frac{p(n)-q(n)}{2}.$$

Proof. The signed class generating function is

$$\sum_{n\geq 0} \left(\sum_{\mu \vdash n} \varepsilon(\mu) \right) x^n = \prod_{j \text{ odd}} \frac{1}{1-x^j} \prod_{j \text{ even}} \frac{1}{1+x^j}.$$

Using $(1+x^{2r})^{-1} = (1-x^{2r})/(1-x^{4r})$ and cancelling the even Euler factors gives

$$\prod_{r\geq 1} (1+x^{2r-1}),$$

the generating function for partitions into distinct odd parts. If the sign row has P_n positive and M_n negative entries, then $P_n + M_n = p(n)$ and $P_n - M_n = q(n)$. □

For a fixed row λ, let $E^-(\lambda)$ be the number of negative values on $\mathcal{E}_n$, and define $O^-(\lambda), O^0(\lambda), O^+(\lambda)$ analogously on $\mathcal{O}_n$.

Proposition 2.2 (Parity balance). *For every $\lambda \vdash n$,*

$$N_n^-(\lambda') - N_n^-((1^n)) = E^-(\lambda) - O^-(\lambda) - O^0(\lambda).$$

Consequently OPAC-038 at degree n is equivalent to

$$E^-(\lambda) \le O^-(\lambda) + O^0(\lambda) \qquad (\lambda \vdash n). \tag{2.1}$$

Proof. The sign twist gives $\chi^{\lambda'}(\mu) = \varepsilon(\mu)\chi^{\lambda}(\mu)$, hence

$$N_n^-(\lambda') = E^-(\lambda) + O^+(\lambda).$$

The sign row is negative on every odd class, so

$$N_n^-((1^n)) = O^-(\lambda) + O^0(\lambda) + O^+(\lambda).$$

Subtract. □

The zero term in (2.1) is structural. In S_3, the row $\chi^{(2,1)}$ has values $2, 0, -1$, and therefore $E^- = 1$, $O^- = 0$, $O^0 = 1$. Thus the stronger inequality $E^- \le O^-$ is false even though parity balance is sharp.

3. Balanced hooks and binary compression

For $0 \le r \le n-1$, write

$$H_r^{(n)} = \chi^{(n-r,1^r)}.$$

When the ambient degree is clear we write simply $H_r(\mu)$.

Proposition 3.1 (Hook generating function). *For every $\mu = (\mu_1, \dots, \mu_\ell) \vdash n$,*

$$\sum_{r=0}^{n-1} H_r^{(n)}(\mu) t^r = \frac{1}{1+t} \prod_{i=1}^{\ell} (1 - (-t)^{\mu_i}). \tag{3.1}$$

Moreover

$$H_r^{(n)}(\mu) = \varepsilon(\mu) H_{n-1-r}^{(n)}(\mu). \tag{3.2}$$

Proof. This is the standard exterior-power form of the Murnaghan–Nakayama hook formula; see, for example, [2, 4]. The symmetry (3.2) is the conjugation identity $(n-r, 1^r)' = (r+1, 1^{n-r-1})$ together with sign twisting. □

Set

$$\lambda_k = (k+1, 1^{k-1}) \vdash 2k, \qquad X_k(\mu) = \chi^{\lambda_k}(\mu) = H_{k-1}^{(2k)}(\mu).$$

Proposition 3.2 (Multiplicity-free middle interpretation). *Let*

$$\Theta_k = \mathrm{Ind}_{S_k \times S_k}^{S_{2k}} (\mathrm{sgn} \boxtimes 1).$$

Then

$$\mathrm{ch}(\Theta_k) = e_k h_k = s_{(k+1,1^{k-1})} + s_{(k,1^k)},$$

and therefore

$$\Theta_k(\mu) = (1 + \varepsilon(\mu)) X_k(\mu).$$

Proof. Under the Frobenius characteristic map, induction corresponds to multiplication, so $\mathrm{ch}(\Theta_k) = e_k h_k$. Pieri's rule gives exactly the two displayed Schur summands. They index conjugate partitions, so the character identity follows from sign twisting. □

The binary geometry is governed by the elementary split–merge move

$$(m, m) \longleftrightarrow (2m).$$

Write every positive integer uniquely as $2^j b$ with b odd. For each odd b, the multiplicities of $b, 2b, 4b, \ldots$ form the b-chain. Repeatedly merging equal parts gives the usual Glaisher correspondence. Fix once and for all a canonical choice of the first active chain and the first active level. This pairs every non-singleton Glaisher state by one split–merge edge; the fixed points are precisely the distinct-odd partitions.

For the arguments below we need only the terminal odd edges, for which the canonical condition has a simple form.

Definition 3.3 (Terminal odd m-edge)**.** Let m be odd. A pair

$$A = \tau \cup (m, m), \qquad B = \tau \cup (2m) \tag{3.3}$$

is called a *terminal odd m-edge* if every odd-base chain $c < m$ is already Glaisher-trivial in τ, and the m-chain of τ contains no part $2^j m$ with $j \geq 1$.

Thus every even part occurring in the common remainder τ of a terminal odd m-edge has odd base strictly larger than m.

Lemma 3.4 (All-odd middle values)**.** *If $\mu \vdash 2k$ has only odd parts, then $X_k(\mu) \geq 0$.*

Proof. Put $P(t) = \prod_i (1 + t^{\mu_i})$. Since $|\mu| = 2k$ and all parts are odd, $\ell(\mu)$ is even; hence $P(-1) = 0$ and

$$Q(t) = \frac{P(t)}{1 + t}$$

is a polynomial of degree $2k - 1$. It is palindromic, so its coefficients at t^{k-1} and t^k agree. Since $(1 + t)Q = P$, twice that common coefficient is $[t^k]P \geq 0$. By (3.1), the coefficient at t^{k-1} is $X_k(\mu)$. □

Because the fixed Glaisher states are distinct-odd, Theorem 3.4 shows that they never create a negative contribution to parity balance. On every paired edge there is exactly one even and one odd conjugacy class. Thus a pair with two negative endpoint values contributes $+1$ to the direct-row deficit $E^- - O^+ - O^0$, a pair with two nonnegative endpoint values contributes -1, and a mixed pair contributes 0.

Proposition 3.5 (Binary edge deficit)**.** *Let N_{--} and N_{++} denote the numbers of canonical Glaisher edges for which both balanced-hook endpoint values are negative and nonnegative, respectively. Then*

$$E^-(\lambda_k) - O^+(\lambda_k) - O^0(\lambda_k) = N_{--} - N_{++}.$$

In particular, any injection from the set of $--$ edges into the set of $++$ edges proves parity balance for the corresponding collection of canonical edges.

4. The exact two-scale formula

Let (3.3) be a canonical edge for the balanced hook, and put

$$q = k - m, \qquad d = k - 2m, \qquad |\tau| = 2q.$$

For any partition ρ, write $H_j(\rho)$ for the hook coefficient of degree j in (3.1), with $H_j(\rho) = 0$ outside $0 \leq j \leq |\rho| - 1$.

Theorem 4.1 (Two-scale normal form)**.** *For the edge* (3.3),

$$X_k(A) = \varepsilon(\tau) H_d(\tau) - 2(-1)^m H_{q-1}(\tau) + H_{d-1}(\tau), \tag{4.1}$$

$$X_k(B) = \varepsilon(\tau) H_d(\tau) - H_{d-1}(\tau). \tag{4.2}$$

Proof. From (3.1),

$$F_A(t) = F_\tau(t)(1-(-t)^m)^2, \qquad F_B(t) = F_\tau(t)(1-t^{2m}).$$

Extracting the coefficient of t^{k-1} gives

$$X_k(A) = H_{k-1}(\tau) - 2(-1)^m H_{q-1}(\tau) + H_{d-1}(\tau),$$
$$X_k(B) = H_{k-1}(\tau) - H_{d-1}(\tau).$$

Since $|\tau| = 2q$, the complement of $k-1$ in the degree $2q-1$ hook polynomial is d. Apply (3.2). ☐

For odd m, the middle term in (4.1) has sign $+2H_{q-1}(\tau)$.

5. A direct theorem for every terminal odd scale $m > k/3$

The next result is the cleanest pointwise theorem currently available in the balanced-hook problem.

Theorem 5.1 (Terminal odd one-third theorem)**.** *Let m be odd and let* (3.3) *be a terminal odd m-edge for λ_k. If*

$$m > d = k - 2m, \qquad \textit{equivalently} \qquad m > \frac{k}{3},$$

then every $--$ edge is mapped injectively to a strict $++$ edge at the same m-scale. Consequently the contribution of all terminal odd m-edges with $m > k/3$ to $N_{--} - N_{++}$ is nonpositive.

Proof. Since $|\tau| = 2(m+d) < 4m$, and every even part of a terminal odd common remainder has odd base $> m$, every even part is $> 2m$. Hence τ contains at most one even part. A level-≥ 2 even part would exceed $4m$, so any even part must be of the form $2b$ with $b > m$ odd.

If τ has no even part, then it is all odd and Theorem 3.4, applied to the two-scale form, gives a nonnegative endpoint. Thus a $--$ source must have

$$\tau^- = \sigma \cup (2b),$$

where σ is all odd. Write $|\sigma| = 2a$; then $a + b = q = m + d$. Put

$$F(t) = \frac{\prod_{x\in\sigma}(1+t^x)}{1+t} = \sum_j f_j t^j.$$

Because $b > m > d$, one has $a = q - b < d < m$, hence $d - b < 0$ and $u := a - b = q - 2b < 0$. Set

$$s = 2a - d = k - 2b.$$

Palindromicity of F gives

$$f_d = f_{s-1}, \qquad f_{d-1} = f_s.$$

The one-even factor is $1 - t^{2b}$, and $2b > d$, so Theorem 4.1 reduces to

$$(X_k(A^-), X_k(B^-)) = (f_s - f_{s-1}, -(f_{s-1} + f_s)).$$

Thus for a $--$ source

$$D := f_{s-1} - f_s > 0, \qquad P := f_{s-1} + f_s > 0,$$

and its endpoint pair is $(-D, -P)$.

Now split the unique higher even part:

$$\tau^+ = \sigma \cup (b, b).$$

This target is all odd and is canonical at the same terminal m-scale. Since $b > d$, its low-degree endpoint difference is

$$X_k(B^+) = f_d - f_{d-1} = D.$$

For the other endpoint, the coefficient of degree d in $\prod_{x\in\sigma}(1+t^x)(1+t^b)^2$ is P, because $b > d$. Moreover

$$H_{q-1}(\tau^+) = 2f_{a-1}.$$

The all-odd polynomial F is palindromic of degree $2a-1$, hence $f_{a-1} = f_a =: C$. Since

$$2C = [t^a] \prod_{x\in\sigma}(1+t^x),$$

we have $C \geq 0$. Therefore

$$(X_k(A^+), X_k(B^+)) = (P + 4C, D),$$

which is strictly positive in both coordinates.

The map is injective: b is recovered as the repeated higher odd part of τ^+, and deleting the two copies recovers σ. □

Corollary 5.2. *Every stable terminal odd scale $m > 2d$ is closed by the same direct split, for every d and every residual length.*

The importance of Theorem 5.1 is that it removes all reservoir estimates from the stable problem. The previously observed Hall obstructions for stronger proper-subpartition target rules remain valid counterexamples to those stronger rules, but they are no longer obstructions to parity balance itself.

6. The first transition band $k/4 < m \leq k/3$

We now enter the first range in which two higher even parts can coexist. Write

$$m \leq d < 2m, \qquad \text{equivalently} \qquad \frac{k}{4} < m \leq \frac{k}{3}.$$

Then

$$|\tau| = 2(m+d) < 6m.$$

The next theorem gives a complete structural classification of strict $--$ sources in this band.

Theorem 6.1 (First-band even-part classification)**.** *Let m be odd and let* (3.3) *be a terminal odd m-edge with $m \leq d < 2m$. If both endpoint values are negative, then exactly one of the following holds:*

(a) $\tau = \sigma \cup (2b)$, *where $b > m$ is odd and σ is a nonempty all-odd multiset;*
(b) $\tau = (2b, 2b)$, *necessarily with $q = 2b$, and the endpoint pair is $(-2,-2)$.*

No other multi-even or higher-level even configuration can be $--$.

Proof. Every even part of τ has odd base $> m$, hence is $> 2m > d$.

Suppose first that τ contains at least two even parts and at least one odd part. Let W be the total weight of the odd parts. Since the two even parts contribute more than $4m$,

$$W < 2(m+d) - 4m = 2(d-m) < d.$$

The odd factors contribute a polynomial

$$\frac{\prod_{x \text{ odd part of } \tau}(1+t^x)}{1+t}$$

of degree $W - 1 < d-1$, while every even factor is $1 - t^e$ with $e > d$. Hence

$$H_{d-1}(\tau) = H_d(\tau) = 0,$$

and (4.2) gives $X_k(B) = 0$, contradicting strict negativity.

Now suppose τ has at least two even parts and no odd part. Since $|\tau| < 6m$ and each even part exceeds $2m$, there are exactly two even parts. Neither can have binary level at least 2, because such a part would exceed $4m$ and leave insufficient weight for the other. Thus

$$\tau = (2b, 2c), \qquad b, c > m \text{ odd}.$$

For $r \leq d < 2m < 2b, 2c$, the relevant hook coefficients are those of $(1+t)^{-1}$:

$$H_d(\tau) = (-1)^d, \qquad H_{d-1}(\tau) = (-1)^{d-1}.$$

Since $\varepsilon(\tau) = 1$,

$$X_k(B) = 2(-1)^d.$$

Thus $B < 0$ forces d odd. Then $q = m + d$ is even and

$$X_k(A) = 2H_{q-1}(\tau).$$

The double product t^{2b+2c} lies above degree $q - 1$, so

$$H_{q-1}(\tau) = (-1)^{q-1}(1 - N) = N - 1,$$

where N is the number of parts among $2b, 2c$ not exceeding $q - 1$. For $A < 0$ we need $N = 0$, so both $2b$ and $2c$ exceed $q - 1$. But $2b + 2c = 2q$, forcing $2b = 2c = q$. This is case (b), and then $(A, B) = (-2, -2)$.

Finally suppose there is exactly one even part. If its binary level is at least 2, it exceeds $4m$. If odd parts remain, their total weight is again less than $2(d - m) < d$, so $H_{d-1} = H_d = 0$ and $B = 0$. If no odd part remains, then τ consists of a single even part of size $2q$, and direct substitution in (4.2) again gives $B = 0$. Therefore any strict one-even source has binary level exactly 1, namely $2b$ with $b > m$ odd. The odd residual cannot be empty, since that single-even case was just excluded. □

The exceptional family is paid exactly.

Proposition 6.2 (Pure double-even payment)**.** *In case* (b) *of Theorem 6.1, the map*

$$(2b, 2b) \longmapsto (2b, b, b)$$

produces a canonical target whose endpoint pair is $(0, 0)$. *These targets are injective in* b.

Proof. For a strict source in case (b), the preceding proof gives $q = 2b$ and d odd. For

$$\tau^+ = (2b, b, b),$$

the factor $1 - t^{2b}$ lies above degrees d and $d - 1$, while

$$\frac{(1 + t^b)^2}{1 + t}$$

has low coefficients $H_d = -(-1)^d$ and $H_{d-1} = (-1)^d$. Since $\varepsilon(\tau^+) = -1$, (4.2) gives $B = 0$. Also $H_{q-1} = 1$, and (4.1) gives $A = 2 + 2(-1)^d = 0$ because d is odd. □

Thus the entire unresolved first band is reduced to the one-even family

$$\tau^- = \sigma \cup (2b).$$

7. The one-even normal form

Let $\tau^- = \sigma \cup (2b)$ be a one-even source in the first band, with $b > m$ odd and σ all odd. Write

$$|\sigma| = 2a, \qquad a + b = q = m + d,$$

and set

$$F(t) = \frac{\prod_{x \in \sigma}(1 + t^x)}{1 + t} = \sum_j f_j t^j. \tag{7.1}$$

Since $2b > d$, the even factor does not alter H_d or H_{d-1}. Define

$$u = a - b = q - 2b, \qquad s = 2a - d = m + u = k - 2b.$$

Proposition 7.1 (One-even coefficient normal form)**.** *For the source* $\tau^- = \sigma \cup (2b)$,

$$X_k(B^-) = -(f_{s-1} + f_s) = -[t^s] \prod_{x \in \sigma}(1 + t^x), \tag{7.2}$$

$$X_k(A^-) = (f_s - f_{s-1}) + 2(f_u - f_{u-1}). \tag{7.3}$$

Here $f_j = 0$ *outside the coefficient range.*

Proof. The polynomial F has degree $2a-1$ and is palindromic. Hence

$$f_d = f_{s-1}, \qquad f_{d-1} = f_s, \qquad f_{q-1} = f_u, \qquad f_{q-1-2b} = f_{u-1}.$$

The one even part gives $\varepsilon(\tau^-) = -1$ and contributes the factor $1 - t^{2b}$. Substitute these identities into Theorem 4.1. Finally, $(1+t)F = \prod_{x\in\sigma}(1+t^x)$ gives (7.2). □

Thus $B^- < 0$ is not an opaque sign condition: it is exactly the existence of a labelled submultiset of σ of total weight s.

7.1. **Odd-split targets.** Allow an odd decomposition

$$2b = v + w, \qquad v, w \text{ odd},$$

and put

$$\tau_{v,w} = \sigma \cup (v, w).$$

The target is all odd. Let $\Delta f_j = f_j - f_{j-1}$.

Proposition 7.2 (Odd-split criterion)**.** *For every canonical odd split $2b = v + w$, one endpoint of the target edge is automatically nonnegative. The other is*

$$B_{v,w} = \Delta f_d + \Delta f_{d-v} + \Delta f_{d-w}. \tag{7.4}$$

Consequently $B_{v,w} \geq 0$ is sufficient for $\tau_{v,w}$ to be a ++ target.

Proof. Since $2b > d$, the term involving f_{d-2b} vanishes. Expanding

$$F(t)(1+t^v)(1+t^w)$$

shows that $H_d - H_{d-1}$ is exactly (7.4). The other endpoint is

$$(H_d + H_{d-1}) + 2H_{q-1}.$$

For an all-odd partition, $H_d + H_{d-1}$ is a subset coefficient and H_{q-1} is the central coefficient of a palindromic quotient, so both are nonnegative. □

A universal odd-split theorem would be attractive, but it is false.

Example 7.3 (Odd splits can all fail)**.** Take

$$m = 11, \quad d = 21, \quad k = 43, \quad b = 13, \qquad \sigma = (17, 15, 5, 1).$$

Then

$$\tau^- = (26, 17, 15, 5, 1)$$

has endpoint pair $(-1,-1)$. The canonical odd splits of 26 give

$$\begin{aligned} (3,23) &\longmapsto (4,-2), & (7,19) &\longmapsto (3,-1),\\ (9,17) &\longmapsto (5,-1), & (11,15) &\longmapsto (6,-2),\\ (13,13) &\longmapsto (3,-1). \end{aligned}$$

Thus no canonical odd split is a ++ target.

The values in Theorem 7.3 follow either from (7.4) or directly from (3.1); they are included in the exact verification script described in Section 10.

7.2. **A universal zero-absorption target.** Odd splits fail pointwise, but the one-even normal form contains enough unused mass to construct a different target family.

Theorem 7.4 (Zero absorption)**.** *Let $\tau^- = \sigma \cup (2b)$ be any strict one-even $--$ source in the first band. Let $R \subseteq \sigma$ be an odd-cardinality submultiset with*

$$0 < \mathrm{wt}(R) < d.$$

Put

$$Z_R = R \cup (2q - \mathrm{wt}(R)). \tag{7.5}$$

Then Z_R is a canonical all-odd common remainder and its two balanced-hook endpoint values are both zero.

Moreover every strict one-even source admits at least one such R.

Proof. Write $w = \mathrm{wt}(R)$ and $L = 2q - w$. Since R has odd cardinality and odd parts, w is odd, so L is odd. Also

$$L > 2q - d = 2m + d = k.$$

Thus the new giant part lies above all three coefficient indices $d-1, d, q-1$.

Because R is nonempty and consists of odd parts,

$$Q_R(t) = \frac{\prod_{x\in R}(1+t^x)}{1+t}$$

is a polynomial of degree $w - 1 < d - 1$. Hence in

$$Q_{Z_R}(t) = (1 + t^L)Q_R(t)$$

we have

$$H_{d-1}(Z_R) = H_d(Z_R) = H_{q-1}(Z_R) = 0.$$

Since Z_R is all odd, $\varepsilon(Z_R) = 1$, and Theorem 4.1 gives endpoint pair $(0, 0)$.

It remains to show existence. Since $b > m$ and $a + b = q = m + d$, we have $a < d$, so

$$\mathrm{wt}(\sigma) = 2a < 2d.$$

The odd residual σ is nonempty and has even cardinality. Therefore at least one part $x \in \sigma$ is $< d$; otherwise its total weight would be at least $2d$. The singleton $R = (x)$ has odd cardinality and satisfies the required inequality. ☐

Corollary 7.5 (Local target existence in the entire first band)**.** *Every strict $--$ terminal odd edge with $k/4 < m \le k/3$ has at least one explicit $++$ target. The pure double-even family uses Theorem 6.2; every other source uses Theorem 7.4.*

The corollary does *not* prove the first-band parity deficit, because different sources may demand the same zero target.

8. The hybrid Hall frontier

The preceding results isolate the unresolved first-band problem as a collision question. Two natural target spaces are available:

- $\mathcal{T}_{\mathrm{split}}$: favorable all-odd targets obtained by canonical odd splits $2b = v + w$ with (7.4) nonnegative;
- $\mathcal{T}_{\mathrm{abs}}$: all-odd zero targets (7.5).

They are automatically disjoint. Indeed, a strict one-even source has $2b \le k$ by (7.2); also $\mathrm{wt}(\sigma) < 2d < k$. Hence every odd-split target has all parts below k, whereas every absorption target has its giant part $L > k$.

The pure double-even targets from Theorem 6.2 contain an even part and are disjoint from both all-odd spaces.

Neither target space is sufficient by itself.

Proposition 8.1 (Absorption-only Hall failure)**.** *At* $(m,d)=(11,20)$ *there are* 27 *one-even* $--$ *sources, but the union of all their zero-absorption targets contains only* 24 *distinct partitions. Thus no absorption-only matching can saturate that band.*

This is an exact finite statement verified by exhaustive integer arithmetic; see Section 10. Together with Theorem 7.3, it shows that both single-reservoir strengthening attempts are genuinely false.

The surviving statement is the following.

Conjecture 8.2 (Two-tier first-band Hall conjecture)**.** *Fix odd* m *and* $m \le d < 2m$. *Partition the strict one-even* $--$ *sources into two classes:*

(i) *split-capable sources, for which at least one canonical odd split satisfies* (7.4)*;*
(ii) *split-dead sources, for which every canonical odd split fails.*

Then the split-capable sources admit a matching into $\mathcal{T}_{\mathrm{split}}$*, and the split-dead sources admit a matching into* $\mathcal{T}_{\mathrm{abs}}$*.*

Because the target spaces are disjoint, the two Hall statements in Theorem 8.2 can be proved independently.

Proposition 8.3 (Finite verification of the hybrid architecture)**.** *For every odd* $m \le 23$ *and every* $m \le d < 2m$*, the conclusion of Theorem 8.2 holds in exact computation. Across these bands there are* 14,817 *split-capable one-even* $--$ *sources in* 138 *nonempty bands and* 20 *split-dead sources in* 13 *bands. Every tested split graph and every tested absorption graph has a matching saturating its source side. The minimum favorable split degree is* 1*, and the minimum absorption degree among split-dead sources is* 4*.*

The largest tested split-capable band is $(m,d)=(23,45)$*, with* 1,234 *sources and* 6,256 *distinct favorable split targets.*

The rarity of split-dead sources is suggestive but is not used as evidence for a theorem. A plausible analytic strategy is to first classify the coefficient patterns that force all quantities in (7.4) negative, and only then prove Hall in the much smaller absorption graph.

9. Even 2-adic scales and a binary-square identity

The preceding theorems concern terminal odd scales. Even scales cannot be treated by independent scale positivity: exact examples have $N_{--}(k,m) > N_{++}(k,m)$. The correct algebraic unit is therefore a binary square rather than a single edge.

Let $X_r(\mu) = H_r^{(|\mu|)}(\mu)$ for a general hook character.

Lemma 9.1 (One-scale Hadamard descent)**.** *For every partition* ρ *and positive integer* a*,*

$$X_r(\rho \cup (a,a)) - X_r(\rho \cup (2a)) = 2(-1)^{a+1} X_{r-a}(\rho \cup a). \tag{9.1}$$

Proof. In the hook generating function, put $s = (-t)^a$. The difference of the two edge factors is

$$(1-s)^2 - (1-s^2) = -2s(1-s).$$

Multiplying by the generating function of ρ and extracting t^r gives (9.1). ☐

Applying Theorem 9.1 twice gives the exact mixed second difference.

Theorem 9.2 (Binary-square descent)**.** *For all positive integers* a, b*,*

$$\begin{aligned} &X_r(\rho \cup (a,a,b,b)) - X_r(\rho \cup (2a,b,b)) \\ &\quad - X_r(\rho \cup (a,a,2b)) + X_r(\rho \cup (2a,2b)) \\ &\qquad\qquad = 4(-1)^{a+b} X_{r-a-b}(\rho \cup a \cup b). \end{aligned} \tag{9.2}$$

Proof. Take the a-edge difference in the first two terms and in the last two terms, apply (9.1), and then take the b-edge difference in the resulting lower-degree pair. ☐

Identity (9.2) explains why source-by-source descent is too rigid at even scales: the cancellation lives on a two-dimensional binary cell. In exact data, every known positive individual-scale deficit is accompanied by a much larger favorable deficit at the adjacent half-scale, but the obvious pointwise half-scale injection fails. The remaining even-scale theorem must therefore be capacitated or telescoping across binary squares or longer 2-adic chains.

10. Computation and independent checking

No computational statement is used as a premise of any theorem above. The finite calculations serve three purposes: they calibrate the canonical normalization, falsify over-strong matching conjectures, and test the surviving Hall architecture.

The principal exact certificate for Theorems 7.3, 8.1 and 8.3 is the pure-Python script

`opac_first_band_hybrid_check.py`

with SHA-256

`49a70d3129f6c9f87d726d01f60ab50c16a174d975c2261b8dfacc800a80acf3`.

It uses only exact Python integers. In addition to the Hall computations stated above, it directly enumerates 1,864 complete canonical pairs for odd $m \leq 9$ in the first band and finds 52 strict $--$ pairs: 48 one-even sources and exactly 4 pure double-even exceptions, in agreement with Theorem 6.1. It also performs 2,000 direct source-formula audits against (3.1), 2,000 favorable odd-split endpoint audits, and 376 direct zero-target audits.

Separate exact certificates check the direct terminal split and its $m > k/3$ extension. These computations are supplementary consistency tests, not proof steps.

11. Complementary established sectors and the global frontier

The binary-compression theorem is one piece of the OPAC-038 program. For context, several complementary sectors are already rigorously controlled in prior work [12]:

- every self-conjugate row satisfies OPAC-038 in every degree;
- a uniform $\sqrt{n}\log n$ two-row boundary strip is solved, together with the first three two-row boundary pairs and their conjugates;
- the complete second exterior-hook conjugate pair $(n-2, 1^2)$ and $(3, 1^{n-3})$ is solved in every degree;
- missing-hook and deep-Durfee mechanisms supply large explicit zero reservoirs;
- in the general even-degree final-defect route, any surviving defect has colength

$$c \geq (0.836269804095714 - o(1))n^{2/3};$$

- exact computation verifies the full OPAC-038 inequality through $n = 29$.

None of these statements closes the remaining interfaces. In the balanced-hook sector, the terminal odd region $m > k/3$ is now closed and the first transition band is reduced to Theorem 8.2. Genuinely even 2-adic scales require an aggregate binary-square or chain theorem. Outside the hook sector, the general even-degree problem still requires a sign-conditioned collision theorem for zero capacity, and odd degree requires a parity-compatible same-row transfer. Thus OPAC-038 remains open.

12. Conclusion

The main simplification obtained here is qualitative. The terminal odd balanced-hook obstruction is not a large-residual asymptotic problem. Above the one-third scale, every negative–negative edge has a unique higher even part and one further binary split reverses both endpoint signs exactly. Immediately below one third, the binary geometry is still rigid enough to classify every strict negative–negative source: the only multi-even family is explicit and harmless, while all remaining sources are one-even.

The one-even normal form exposes two complementary target mechanisms. Odd splitting is sensitive to the residual finite-difference profile; zero absorption ignores that profile but creates a giant all-odd marker. Each mechanism alone is too rigid, and exact counterexamples show this sharply. Their target spaces are nevertheless disjoint, leaving a clean two-tier Hall problem rather than a vague global matching problem.

A proof of Theorem 8.2 would close the entire first terminal-odd transition band $k/4 < m \leq k/3$. The next conceptual step beyond that band is already visible in (9.2): even scales must be organized by whole binary squares or chains, not paid independently. These two interfaces are the most direct hook-theoretic obstacles currently separating the proved terminal region from a complete balanced-hook theorem.

Data and code availability

The exact verification script cited in Section 10 is supplied with the source package for this preprint. All finite claims in this paper are explicitly labelled as computational and are not used in proofs of the stated theorems.

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