

PROGRESS ON OPAC-038: BINARY COMPRESSION, SPLIT FUSION, AND UNWEIGHTED CHARACTER INEQUALITIES

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ABSTRACT. For an irreducible character χ^λ of S_n , let $N_n^-(\lambda)$ be the number of conjugacy classes on which χ^λ is negative, with no class-size weighting. Hopkins' OPAC-038 asks whether, for all sufficiently large n , the sign representation maximizes N_n^- . We prove several exact advances toward this problem and isolate the interfaces that remain open. The global results apply to every irreducible row: conjugation modules and Sundaram's Schur-positive class sums yield unweighted moment domination on parity-balanced class reservoirs. A complementary exact case study treats balanced hooks under Glaisher compression: we derive a closed formula across a canonical Glaisher edge, prove an injective positivity theorem above the threshold $m > k/3$, classify all strict negative sources in the next range $k/4 < m \leq k/3$, construct parity-complete all-odd targets, and exhibit an infinite family for which every canonical two-part odd refinement fails. On the exceptional distinct-odd sector, restriction to A_n gives exact split-fusion multiplicities

$$a_{ij} = \frac{g(\lambda, \gamma_i, \gamma_j) + \chi^\lambda(\delta_i)\delta_{ij}}{2}, \quad b_{ij} = \frac{g(\lambda, \gamma_i, \gamma_j) - \chi^\lambda(\delta_i)\delta_{ij}}{2},$$

and hence $g(\lambda, \gamma_i, \gamma_j) \geq |\chi^\lambda(\delta_i)|$. We then derive OPAC-specific unweighted moment inequalities from conjugation modules and Sundaram's Schur-positive class sums, including $B_j \geq |O_j|$ for every $j \geq 0$. Matrix and plethystic constructions are retained as supporting infrastructure, but are explicitly separated from the paper's novelty claims: character-table diagonalization, Parseval, and positivity of multiplication matrices are standard consequences of established theory. Finally, we prove no-go statements excluding several tempting scalar, matching, and generic-matrix shortcuts. We end with a precise account of what remains open; the proposed coupling is presented only as a research programme, not as a theorem or a reduction of OPAC-038. OPAC-038 remains open.

1. INTRODUCTION AND EXACT FORMULATIONS

Let $p(n)$ be the partition number. The irreducible characters of S_n and its conjugacy classes are both indexed by partitions of n ; we write $\chi^\lambda(\mu)$ for the value in row λ and column μ . Define

$$N_n^-(\lambda) = \#\{\mu \vdash n : \chi^\lambda(\mu) < 0\}.$$

Hopkins' OPAC-038 asks whether, for all sufficiently large n , the sign representation (1^n) maximizes this statistic [1]. The problem is genuinely unweighted: each conjugacy class contributes one unit, independently of its size or centralizer.

Let $q(n)$ be the number of self-conjugate partitions, equivalently the number of partitions into distinct odd parts. The sign character is negative precisely on odd permutations, and

Date: 27 September 2026.

2020 Mathematics Subject Classification. 20C30, 05E10, 05A17, 20C15.

Key words and phrases. symmetric group, character table, negative character values, Kronecker coefficients, alternating group, Glaisher correspondence, plethysm, Murnaghan–Nakayama rule, conjugation representation, Schur positivity.

Euler's product identity gives

$$N_n^-((1^n)) = r_n := \frac{p(n) - q(n)}{2}.$$

Thus OPAC-038 asks for $N_n^-(\lambda) \leq r_n$ eventually. All results stated as theorems below are proved symbolically in the manuscript; no finite computation is used as a premise.

The purpose of this paper is to collect the strongest exact advances obtained so far into one proof architecture. The contributions are not a solution. The principal new mathematical content is concentrated in three places: the balanced-hook compression theorems of Section 3, the exact A_n split-fusion multiplicity formula of Section 4, and the OPAC-specific unweighted moment domination of Section 6. Sections 5 and 7 record supporting transforms and matrix realizations because they clarify how these results interact, but we do *not* claim the underlying Fourier inversion, Parseval identity, character-table diagonalization, or the elementary fact that multiplication by a genuine character has nonnegative integral structure constants as new results. This distinction is important: the aim is to identify what has actually advanced OPAC-038, not to rename standard representation-theoretic machinery.

There are several equivalent or near-equivalent viewpoints. Let $K_\lambda = (g(\lambda, \mu, \nu))_{\mu, \nu \vdash n}$ be multiplication by $[\lambda]$ in the representation ring. Character-table diagonalization gives eigenvalues $\chi^\lambda(\rho)$, $\rho \vdash n$. Hence

$$N_n^-(\lambda) = n_-(K_\lambda),$$

and the target is the inertia bound $n_-(K_\lambda) \leq r_n$. Let J be partition conjugation on the irreducible basis. Since $K_{\lambda'} = JK_\lambda = K_\lambda J$, one obtains a J -block formulation. On the class side, the Glaisher correspondence decomposes partitions into parity-balanced binary components plus $q(n)$ distinct-odd singleton components. This gives the deficit form

$$(1.1) \quad D_\lambda \leq \sum_C (h(C) - N_C(\lambda)),$$

where D_λ counts negative values on distinct-odd singleton classes, C ranges over nontrivial balanced components, $h(C) = |C|/2$, and $N_C(\lambda)$ is the number of negative values in C . Equation (1.1) makes the global difficulty transparent: exceptional negative values must be paid for by unused negative capacity spread across balanced components.

Main results and logical status. The paper has three theorem-level advances that feed directly into the OPAC programme, two of which are global in the row parameter. First, the module argument of Section 6 gives, for every $\lambda \vdash n$, the genuinely unweighted inequalities

$$E_j \geq |O_j|, \quad B_j \geq |O_j| \quad (j \geq 0),$$

with $|B| = |O| = r_n$; the second inequality is localized to the exact nonexceptional parity-balanced reservoirs. Second, Theorem 4.2 resolves the two A_n fusion channels exactly for every non-self-conjugate multiplier and upgrades principal-hook nonvanishing to a multiplicity inequality. Third, Theorems 3.3–3.8 give an exact Glaisher-compression theory for the balanced hooks, including an injective positive payment above $m > k/3$, complete classification in the next range, parity-complete absorption, and an infinite obstruction to any universal two-part odd-refinement rule.

None of these statements proves OPAC-038. The moment inequalities are global but do not determine negative support; the split-fusion theorem controls exceptional distinct-odd columns but not their global capacity; and the balanced-hook theorems concern one infinite family of rows. Sections 7–9 make these limitations explicit and formulate the remaining coupling problem. Equal-part plethystic coarsening and binary-square descent are included later as supporting exact identities, not as claimed solutions of either interface.

2. PROVENANCE AND NOVELTY BOUNDARY

Because several ingredients below are classical or follow formally from established theorems, we record the provenance boundary before giving the proofs. This also prevents useful infrastructure from being mistaken for the main mathematical contribution.

- (i) The Murnaghan–Nakayama rule, Frobenius characteristic map, character orthogonality, Clifford theory for $A_n \triangleleft S_n$, and character-table diagonalization of representation-ring multiplication are standard; see [5, 7, 11].
- (ii) The plethystic Murnaghan–Nakayama rule is prior work, including Wildon’s combinatorial proof and the later determinant/vertex-operator formulation of Cao–Jing–Liu; for the latter we cite the publicly available arXiv version [13, 4]. Accordingly, Section 5 uses the coarsening transform only as supporting information. Its Fourier inversion and Parseval statements are not presented as independent breakthroughs.
- (iii) Sundaram proved the Schur positivity of the class sums used in Sections 6–7 [12]. Once such a genuine character Φ_T is available, entrywise nonnegativity of its multiplication matrix and the corresponding spectral selector formula are formal. Our use of these modules is OPAC-specific: tensor powers cancel centralizer weights and produce the infinite unweighted inequalities in Section 6. The constituent-level formulas in Section 7 are retained because they preserve information discarded by scalar moments, not because matrix multiplication positivity itself is novel.
- (iv) Pak–Panova–Vallejo already use the A_n splitting mechanism to deduce principal-hook nonvanishing for Kronecker coefficients [9]. The new point in Section 4 is the simultaneous retention of both split fusion channels, giving their exact sum and difference and hence the multiplicity-level formulas (4.2)–(4.3).
- (v) The balanced-hook theorems in Section 3 are proved directly here from the hook generating function and Glaisher compression. Their role is not merely illustrative: they supply an infinite, nontrivial family in which OPAC’s local capacity problem can be attacked exactly, and they expose both successful payment mechanisms and provable obstructions to overly local rules.

No theorem below uses computational evidence as a premise. To keep the archival article independently auditable, numerical ranges whose scripts and raw outputs are not deposited with this version are not asserted as results.

3. PARITY BALANCE AND GLAISHER COMPRESSION

For $\mu \vdash n$ put $\varepsilon(\mu) = (-1)^{n-\ell(\mu)}$. Let E_n and O_n be the even and odd conjugacy classes.

Proposition 3.1 (Sign-row count). *For every n ,*

$$\sum_{\mu \vdash n} \varepsilon(\mu) = q(n), \quad N_n^-(1^n) = \frac{p(n) - q(n)}{2}.$$

Proof. The signed class generating function is

$$\prod_{j \text{ odd}} (1 - x^j)^{-1} \prod_{j \text{ even}} (1 + x^j)^{-1} = \prod_{r \geq 1} (1 + x^{2r-1}),$$

the generating function for distinct-odd partitions. If P_n, M_n are the positive and negative sign-row counts, then $P_n + M_n = p(n)$ and $P_n - M_n = q(n)$. $\square$

For a fixed λ , write $E_-(\lambda)$ for the number of negative values on even classes and O_-, O_0, O_+ analogously on odd classes.

Proposition 3.2 (Parity balance). *For every $\lambda \vdash n$,*

$$N_n^-(\lambda') - N_n^-((1^n)) = E_-(\lambda) - O_-(\lambda) - O_0(\lambda).$$

Consequently the OPAC inequality in degree n is equivalent, after conjugating rows if necessary, to

$$(3.1) \quad E_-(\lambda) \leq O_-(\lambda) + O_0(\lambda).$$

Proof. Use $\chi^{\lambda'}(\mu) = \varepsilon(\mu)\chi^\lambda(\mu)$ and partition the odd classes according to sign. $\square$

The relevant combinatorics comes from Glaisher's operation $(m, m) \leftrightarrow (2m)$. Fix an odd integer b . The multiplicities of the parts $2^j b$ form the b -chain of a partition; replacing two copies of $2^j b$ by one copy of $2^{j+1} b$, or reversing that move, preserves total size. A *Glaisher component* is the set of partitions reachable by these moves independently on all odd-base chains. Every nontrivial component contains equally many even and odd permutations, whereas the singleton components are exactly the partitions into distinct odd parts. Thus the $q(n)$ exceptional classes in Proposition 3.1 are precisely the fixed points of Glaisher compression.

For the balanced hook $\lambda_k = (k+1, 1^{k-1}) \vdash 2k$, we call a pair $A = \tau \cup (m, m)$, $B = \tau \cup (2m)$ with m odd a *canonical odd Glaisher edge*. It is *terminal at the odd scale m* when all smaller odd-base chains are already Glaisher-trivial and the m -chain contains no part $2^j m$ with $j \geq 2$. The interval $m \leq d < 2m$ appearing below is equivalently $k/4 < m \leq k/3$; we refer to it only descriptively as the first range beyond the one-third threshold.

3.1. Balanced hooks: an exact formula across a Glaisher edge. For $0 \leq r \leq n-1$ let $H_r^{(n)} = \chi^{(n-r, 1^r)}$. The exterior-power formula is

$$(3.2) \quad \sum_{r=0}^{n-1} H_r^{(n)}(\mu) t^r = \frac{1}{1+t} \prod_i (1 - (-t)^{\mu_i}),$$

with $H_r^{(n)}(\mu) = \varepsilon(\mu) H_{n-1-r}^{(n)}(\mu)$.

Fix the balanced hook $\lambda_k = (k+1, 1^{k-1}) \vdash 2k$ and write $X_k(\mu) = \chi^{\lambda_k}(\mu)$. Consider a canonical odd-scale edge

$$(1) \quad A = \tau \cup (m, m), \quad B = \tau \cup (2m).$$

with m odd, and put $q = k - m$, $d = k - 2m$, $|\tau| = 2q$. Write $H_j(\tau)$ for the coefficient of t^j in (3.2) for τ .

Theorem 3.3 (Hook values across a canonical Glaisher edge). *For the edge (1),*

$$(3.3) \quad X_k(A) = \varepsilon(\tau) H_d(\tau) - 2(-1)^m H_{q-1}(\tau) + H_{d-1}(\tau),$$

$$(3.4) \quad X_k(B) = \varepsilon(\tau) H_d(\tau) - H_{d-1}(\tau).$$

Proof. In (3.2) the two edge factors are $(1 - (-t)^m)^2$ and $(1 - t^{2m})$. Extract t^{k-1} and use complementary hook symmetry in degree $2q - 1$. $\square$

A useful elementary fact is that if $\mu \vdash 2k$ has only odd parts, then $X_k(\mu) \geq 0$: after writing $P(t) = \prod_i (1 + t^{\mu_i})$ and $Q = P/(1+t)$, palindromicity gives equality of the two middle coefficients and hence $2[t^{k-1}]Q = [t^k]P \geq 0$.

Theorem 3.4 (Injective positivity above the one-third threshold). *Let m be odd and suppose the edge (1) is terminal in the sense that all smaller odd-base chains are Glaisher-trivial and the m -chain contains no higher binary part. If $m > d$ (equivalently $m > k/3$), every edge with both endpoint values negative admits an injective same-scale map to an edge with both endpoint values strictly positive.*

Proof. Terminality and $|\tau| = 2(m + d) < 4m$ imply that a negative-negative remainder has the form $\tau^- = \sigma \cup (2b)$ with $b > m$ odd and σ all odd. Put $|\sigma| = 2a$ and

$$F(t) = \frac{\prod_{x \in \sigma} (1 + t^x)}{1 + t} = \sum_j f_j t^j.$$

With $s = 2a - d = k - 2b$, palindromicity yields

$$(X_k(A^-), X_k(B^-)) = (f_s - f_{s-1}, -(f_{s-1} + f_s)) = (-D, -P)$$

for $D, P > 0$. Split the unique higher even part, replacing $2b$ by (b, b) . The resulting all-odd remainder has endpoint pair

$$(P + 4C, D), \quad C = f_{a-1} = f_a \geq 0.$$

It is strictly positive, and the repeated part b recovers the source, so the map is injective. $\square$

3.2. The range $k/4 < m \leq k/3$. Assume $m \leq d < 2m$, equivalently $k/4 < m \leq k/3$. The source geometry remains unexpectedly rigid.

Theorem 3.5 (Classification immediately below the one-third threshold). *Every strict negative-negative terminal odd source is exactly one of:*

- (a) $\tau = \sigma \cup (2b)$ with $b > m$ odd and σ a nonempty all-odd multiset;
- (b) $\tau = (2b, 2b)$, necessarily $q = 2b$, with endpoint pair $(-2, -2)$.

No other multi-even or higher-binary-level configuration is strictly negative at both endpoints.

Proof. If at least two even parts and an odd part occur, the total odd weight is $< 2(d - m) < d$, forcing $H_{d-1} = H_d = 0$ and hence $X_k(B) = 0$. If there are only even parts, size forces exactly two level-one parts; direct evaluation shows negativity only in the equal case. A unique even part of binary level at least two similarly leaves too little odd weight and gives $X_k(B) = 0$. $\square$

The exceptional double-even family is paid injectively by

$$(2b, 2b) \mapsto (2b, b, b),$$

whose endpoint pair is $(0, 0)$. For the one-even family write $|\sigma| = 2a$, $a + b = q$, and set $u = a - b = q - 2b$, $s = 2a - d = m + u$. Then

Proposition 3.6 (One-even finite-difference form). *With $F(t) = \sum f_j t^j$ as above,*

$$(3.5) \quad X_k(B^-) = -(f_{s-1} + f_s), \quad X_k(A^-) = \Delta f_s + 2\Delta f_u,$$

where $\Delta f_j = f_j - f_{j-1}$.

For a canonical odd split $2b = v + w$, v, w odd, one target endpoint is automatically nonnegative and the other is

$$(2) \quad B_{v,w} = \Delta f_d + \Delta f_{d-v} + \Delta f_{d-w}.$$

Odd splitting alone is not universal: at $(m, d, k) = (11, 21, 43)$, $b = 13$, $\sigma = (17, 15, 5, 1)$ gives source endpoints $(-1, -1)$ and every canonical odd split has negative B -endpoint.

Theorem 3.7 (Parity-complete absorption). *Let $\tau^- = \sigma \cup (2b)$ be a strict one-even source in the first band. For every nonempty submultiset $R \subseteq \sigma$ of weight $w < d$, there is an explicit favorable all-odd target.*

- (i) If $|R|$ is odd, $Z_R = R \cup (2q - w)$ has endpoint pair $(0, 0)$.
- (ii) If $|R|$ is even and d is even, $Z_R = R \cup (d + 1, 2q - w - d - 1)$ has pair $(2H_{q-1}(Z_R), 0)$.
- (iii) If $|R|$ is even and d is odd, $Z_R = R \cup (d, 2q - w - d)$ has pair $(1 + 2H_{q-1}(Z_R), 1)$.

In particular every strict one-even source has at least one favorable absorption target.

Proof. The residual quotient from R has degree $w - 1 < d - 1$. The added odd parts are chosen above the low-degree window, except for the controlled part d in case (iii). Substitution into Theorem 3.3 gives the displayed pairs. Nonnegativity of the middle coefficient follows from the all-odd lemma. Since $\text{wt}(\sigma) < 2d$, some singleton $R = (x)$ has $x < d$. $\square$

The following symbolic family shows that failure of two-part odd splitting persists indefinitely.

Theorem 3.8 (Infinite split-dead family). *Let a be odd, $j \geq 1$, $a \geq 4j + 1$, and set*

$$m = 2a + 1, \quad d = 3a + 4j + 2, \quad b = 2a + 2j + 1,$$

$$\sigma = (1, a, 2a + 4j + 1, 3a + 2).$$

Then $m \leq d < 2m$, the source $\sigma \cup (2b)$ has endpoints $(-1, -1)$, and every canonical two-part odd split of $2b$ has strictly negative B -endpoint.

Proof. The part 1 cancels the denominator in F , leaving a product of three binomials. At the relevant indices one gets $(f_s, f_{s-1}) = (1, 0)$ and $(f_u, f_{u-1}) = (0, 1)$, hence $(-1, -1)$. Since d is odd, the correction indices in (2) are even. A nonnegative split would force a split part equal to 1 or d , but each violates canonicity by repeating an existing residual part. $\square$

These results show both abundance and collision. Favorable targets exist sourcewise, while explicit symbolic counterexamples rule out several overly local matching rules. What remains is a genuine global collision-control problem: the present manuscript does not claim a Hall theorem for the union of all split and absorption targets.

4. THE ALTERNATING-GROUP SPLIT-FUSION BRIDGE

Distinct-odd classes are exactly the S_n classes that split in A_n , and they are in bijection with self-conjugate partitions via principal hooks. This identifies the exceptional Glaisher sector with a canonical index-two representation-theoretic sector.

Let $\gamma_1, \dots, \gamma_q$ be the self-conjugate partitions of n , and let δ_i be the distinct-odd principal-hook partition attached to γ_i . Write

$$\text{Res}_{A_n}^{S_n} \chi^{\gamma_i} = \alpha_i + \beta_i, \quad \Delta_i = \alpha_i - \beta_i.$$

For non-self-conjugate λ , $\psi_\lambda = \text{Res}_{A_n}^{S_n} \chi^\lambda$ is irreducible and invariant under the outer involution.

Theorem 4.1 (Difference-character eigenvector identity). *For every i ,*

$$(4.1) \quad \psi_\lambda \Delta_i = \chi^\lambda(\delta_i) \Delta_i.$$

Consequently

$$\langle \psi_\lambda \Delta_i, \Delta_j \rangle = 2\chi^\lambda(\delta_i) \delta_{ij}.$$

Proof. Δ_i is supported only on the two A_n classes lying below the split S_n class δ_i , and takes opposite values on the two halves. Since ψ_λ is the restriction of an S_n class function, it is constant on those halves, with value $\chi^\lambda(\delta_i)$. Pointwise multiplication gives (4.1). $\square$

Define the split-fusion multiplicities

$$a_{ij} = \langle \psi_\lambda \alpha_i, \alpha_j \rangle = \langle \psi_\lambda \beta_i, \beta_j \rangle, \quad b_{ij} = \langle \psi_\lambda \alpha_i, \beta_j \rangle = \langle \psi_\lambda \beta_i, \alpha_j \rangle.$$

Theorem 4.2 (Exact split-fusion identity). *For non-self-conjugate λ ,*

$$(4.2) \quad a_{ij} = \frac{g(\lambda, \gamma_i, \gamma_j) + \chi^\lambda(\delta_i)\delta_{ij}}{2}, \quad b_{ij} = \frac{g(\lambda, \gamma_i, \gamma_j) - \chi^\lambda(\delta_i)\delta_{ij}}{2}.$$

In particular,

$$(4.3) \quad g(\lambda, \gamma_i, \gamma_i) \geq |\chi^\lambda(\delta_i)|, \quad g(\lambda, \gamma_i, \gamma_i) \equiv \chi^\lambda(\delta_i) \pmod{2}.$$

Off the diagonal, $g(\lambda, \gamma_i, \gamma_j)$ is even.

Remark 4.3 (Scope of the split-fusion formula). The hypothesis $\lambda \neq \lambda'$ is essential to the two-channel form above. If λ is self-conjugate then $\text{Res}_{A_n}^{S_n} \chi^\lambda$ itself splits, and the multiplier is exchanged by the outer involution; the natural bookkeeping is therefore four-channel rather than two-channel. No extension of Theorem 4.2 to that case is asserted here.

Proof. Theorem 4.1 gives $a_{ij} - b_{ij} = \chi^\lambda(\delta_i)\delta_{ij}$. Restriction-induction reciprocity across the index-two extension gives $a_{ij} + b_{ij} = g(\lambda, \gamma_i, \gamma_j)$. Solve the two equations. Nonnegativity and integrality of a_{ii}, b_{ii} give (4.3). $\square$

This sharpens the principal-hook positivity mechanism of Pak–Panova–Vallejo [9]: their Main Lemma yields the nonvanishing implication $\chi^\lambda(\delta_i) \neq 0 \Rightarrow g(\lambda, \gamma_i, \gamma_i) > 0$, whereas (4.2) records the exact multiplicity imbalance and hence the magnitude bound. For a negative exceptional value, $b_{ii} \geq a_{ii} + 1$; every negative distinct-odd singleton therefore carries a canonical crossed-channel surplus witness inside A_n .

In matrix form, with $A = (a_{ij})$, $B = (b_{ij})$, $G = (g(\lambda, \gamma_i, \gamma_j))$, and $X = \text{diag}(\chi^\lambda(\delta_i))$,

$$(4.4) \quad A - B = X, \quad A + B = G,$$

so the $2q \times 2q$ split-fusion block is

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix},$$

whose invariant block is G and anti-invariant block is exactly X . Thus the exceptional values are literally the spectrum of multiplication by ψ_λ on the difference-character sector.

A crucial no-go is that (4.4) alone does not imply the needed inertia bound. The generic matrices

$$B = \begin{pmatrix} 4 & 4 \\ 4 & 1 \end{pmatrix}, \quad X = \text{diag}(-1, -1), \quad A = B + X \geq 0$$

satisfy the same entrywise grammar, but $G = 2B + X$ has only one nonnegative eigenvalue. Any successful use of split fusion must exploit additional fusion-ring or class-capacity structure.

5. EQUAL-PART COARSENING AS SUPPORTING FOURIER INFRASTRUCTURE

The Britnell–Wildon theory of cyclic extensions [2, 3] motivates replacing one binary merge by a complete coarsening fibre. Fix $\lambda \vdash n$, integers $k, m \geq 1$, and a residual partition τ with $|\tau| + km = n$. For $\nu \vdash m$ define

$$F(\nu) = \chi^\lambda(\tau \cup k\nu),$$

where $k\nu$ multiplies every part of ν by k .

Proposition 5.1 (Equal-part coarsening transform). *As a class function on S_m ,*

$$(5.1) \quad F(\nu) = \sum_{\alpha \vdash m} c_\alpha \chi^\alpha(\nu), \quad c_\alpha = \langle s_\lambda, p_\tau s_\alpha[p_k] \rangle \in \mathbb{Z}.$$

Equivalently,

$$c_\alpha = \sum_{\rho \vdash m} \frac{\chi^\alpha(\rho) F(\rho)}{z_\rho}.$$

Proof. Use $s_\alpha[p_k] = \sum_{\rho \vdash m} \chi^\alpha(\rho) p_{k\rho}/z_\rho$ and Frobenius orthogonality. $\square$

The proof is deliberately included only to fix normalization. It is Fourier inversion in the irreducible character basis, and no novelty is claimed for that formal step. The word “virtual” is essential: these Fourier coefficients need not be nonnegative. Thus a coarsening fibre cannot be treated as a genuine character without further structure.

The plethystic Murnaghan–Nakayama rule, in the forms developed by Wildon and by Cao–Jing–Liu [13, 4], supplies exactly that further structure for the extremal modes. In the pure case $n = km$,

$$c_+(\lambda; k, m) = \langle s_\lambda, h_m[p_k] \rangle, \quad c_-(\lambda; k, m) = \langle s_\lambda, e_m[p_k] \rangle.$$

The plethystic rule gives $c_+ \in \{0, \pm 1\}$, with nonzero coefficient exactly for the corresponding horizontal k -border-strip condition, and conjugation gives

$$(5.2) \quad c_-(\lambda; k, m) = (-1)^{m(k-1)} c_+(\lambda'; k, m) \in \{0, \pm 1\}.$$

If E and O denote the weighted sums over even and odd coarsenings in S_m , then

$$E = \frac{c_+ + c_-}{2}, \quad O = \frac{c_+ - c_-}{2},$$

so $E, O \in \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$ independently of the size of the fibre.

The stronger residual form is more useful.

Proposition 5.2 (Residual plethystic spectrum). *Let $r = n - km$ and, for $\tau \vdash r$, define*

$$C_+^\lambda(\tau) = \sum_{\nu \vdash m} \frac{\chi^\lambda(\tau \cup k\nu)}{z_\nu}.$$

Then

$$(5.3) \quad C_+^\lambda(\tau) = \sum_{\mu \vdash r} \varepsilon_\mu^\lambda \chi^\mu(\tau), \quad \varepsilon_\mu^\lambda \in \{0, \pm 1\},$$

where ε_μ^λ is the plethystic Murnaghan–Nakayama coefficient of s_λ in $s_\mu h_m[p_k]$.

Proof. Expand $h_m[p_k] = \sum_{\nu \vdash m} p_{k\nu}/z_\nu$, multiply by s_μ , and apply character orthogonality in degree r . The coefficient restriction follows from the plethystic Murnaghan–Nakayama rule. $\square$

Corollary 5.3 (Exact residual Parseval capacity). *Let*

$$A(\lambda; k, m) = \#\{\mu \vdash r : \varepsilon_\mu^\lambda \neq 0\}.$$

Then

$$(5.4) \quad \sum_{\tau \vdash r} \frac{|C_+^\lambda(\tau)|^2}{z_\tau} = A(\lambda; k, m),$$

and $|C_+^\lambda(\tau)| \leq \sqrt{A(\lambda; k, m) z_\tau}$.

This is an exact capacity identity for arbitrary residual cycle type. It is still weighted by $1/z_\tau$, so it does not itself control the unweighted number of negative classes.

There is a second residual mode

$$C_-^\lambda(\tau) = \sum_{\nu \vdash m} \frac{\text{sgn}(\nu) \chi^\lambda(\tau \cup k\nu)}{z_\nu},$$

with the pointwise identity

$$(5.5) \quad C_-^\lambda(\tau) = (-1)^{m(k-1)} \text{sgn}(\tau) C_+^{\lambda'}(\tau).$$

Hence the even/odd coarsening energies are exactly controlled by the two signed border-strip supports. This resolves the residual Fourier structure but not the unweighted OPAC interface.

6. CROSSING THE WEIGHTED-TO-UNWEIGHTED BARRIER

The decisive deweighting observation comes from genuine conjugation representations. Let κ_n be the character of S_n acting on itself by conjugation. At a class μ ,

$$\kappa_n(\mu) = z_\mu.$$

Therefore, for every irreducible $\chi = \chi^\lambda$ and integer $j \geq 0$,

$$(6.1) \quad R_j(\lambda) := \sum_{\mu \vdash n} \chi(\mu)^j = \langle \kappa_n \chi^j, 1 \rangle \in \mathbb{Z}_{\geq 0}.$$

Twisting by sign gives

$$(6.2) \quad T_j(\lambda) := \sum_{\mu \vdash n} \text{sgn}(\mu) \chi(\mu)^j = \langle (\kappa_n \otimes \text{sgn}) \chi^j, 1 \rangle \in \mathbb{Z}_{\geq 0}.$$

If E_j, O_j are the unweighted moments on even and odd classes, then $R_j = E_j + O_j$, $T_j = E_j - O_j$, and hence

$$(6.3) \quad E_j \geq |O_j| \quad (j \geq 0).$$

This is an exact unweighted moment-domination theorem.

Inducing the conjugation action of A_n gives a split-sensitive refinement. If θ_A is the conjugation character of A_n and $\Theta = \text{Ind}_{A_n}^{S_n} \theta_A$, then

$$(6.4) \quad \Theta(\mu) = \begin{cases} 0, & \mu \text{ odd}, \\ z_\mu, & \mu \text{ even non-split}, \\ 2z_\mu, & \mu \text{ distinct odd}. \end{cases}$$

Consequently

$$(6.5) \quad U_j(\lambda) := \sum_{\mu \in B} \chi(\mu)^j + 2 \sum_{\delta \in \text{DO}_n} \chi(\delta)^j = \langle \Theta \chi^j, 1 \rangle \geq 0,$$

where B denotes the even non-split classes.

Sundaram's work on Schur-positive class sums [12] localizes this further. If T is a class subset such that $P_T = \sum_{\mu \in T} p_\mu$ is Schur-positive, there is a genuine character Φ_T with $\Phi_T(\mu) = z_\mu$ on T and 0 off T . Hence

$$(6.6) \quad \sum_{\mu \in T} \chi^\lambda(\mu)^j = \langle \Phi_T(\chi^\lambda)^j, 1 \rangle \geq 0.$$

Sundaram's theorem includes the exact balanced set B of even non-split classes and the set N of all non-distinct-odd classes. Let

$$B_j = \sum_{\mu \in B} \chi(\mu)^j.$$

Since $|B| = |O| = r_n$, the genuine characters Φ_N and $\Phi_N \otimes \text{sgn}$ give, for every $j \geq 0$,

$$B_j + O_j = \langle \Phi_N(\chi^\lambda)^j, 1 \rangle \geq 0, \quad B_j - O_j = \langle (\Phi_N \otimes \text{sgn})(\chi^\lambda)^j, 1 \rangle \geq 0.$$

Hence we obtain the sharper theorem

$$(6.7) \quad B_j \geq |O_j| \quad (j \geq 0).$$

This proof is worth recording explicitly: using the conjugate row λ' would recover the second inequality only for odd j ; the sign twist of the genuine module is what gives both inequalities uniformly for every j . The result compares the two parity-balanced nonexceptional reservoirs of equal cardinality.

However, scalar moments are not enough: the abstract integer multisets $B = \{-1, 3\}$ and $O = \{-1, -1\}$ satisfy $B_j \geq |O_j|$ for all $j \geq 0$ while having an unfavorable negative-count comparison. The missing information must retain constituent labels.

7. MATRIX REALIZATION OF THE BALANCED RESERVOIR

Let T be any class subset with Schur-positive P_T . Let $K(\Phi_T)$ be multiplication by $[\Phi_T]$ in the irreducible basis.

Proposition 7.1 (Class-selector multiplication matrix). *$K(\Phi_T)$ is symmetric, positive semi-definite, entrywise nonnegative and integral, with*

$$(7.1) \quad \text{spec } K(\Phi_T) = \{z_\mu : \mu \in T\} \cup \{0 : \mu \notin T\}, \quad \text{rank } K(\Phi_T) = |T|.$$

Its range is exactly the character-table spectral span indexed by T .

Proof. The representation-ring multiplication matrices are diagonalized by the character table. The eigenvalue of $K(\Phi_T)$ at class μ is $\Phi_T(\mu) = z_\mu 1_T(\mu)$. Entrywise nonnegativity and integrality are tensor-product multiplicities. $\square$

Proposition 7.1 is a matrix restatement of Schur positivity together with standard character-table diagonalization; it is not asserted as a new positivity theorem. Its value here is that for the particular OPAC reservoir B the rank is exactly the conjectured extremal count r_n .

For $T = B$, write

$$(7.2) \quad L_B = K(\Phi_B), \quad \text{rank } L_B = |B| = r_n.$$

This is a canonical PSD matrix of exactly the conjectured extremal rank. For each λ define

$$(7.3) \quad M_B(\lambda) = L_B K_\lambda = K_{\Phi_B \otimes \lambda}.$$

It is symmetric, entrywise nonnegative and integral, with nonzero eigenvalues $z_\mu \chi^\lambda(\mu)$ for $\mu \in B$. Thus the complete sign pattern on the balanced even reservoir is realized as the inertia of a concrete nonnegative representation-ring matrix.

The strongest information appears before taking traces.

Proposition 7.2 (Constituent-level form of the balanced inequalities). *For every $\alpha, \beta, \lambda \vdash n$ and $j \geq 1$,*

$$(7.4) \quad H_j(\alpha, \beta; \lambda) := \sum_{\mu \in B} z_\mu^{j-1} \chi^\lambda(\mu)^j \chi^\alpha(\mu) \chi^\beta(\mu) = (M_B(\lambda)^j)_{\alpha\beta} \in \mathbb{Z}_{\geq 0}.$$

In particular, at $j = 1$ all centralizer weights disappear:

$$(7.5) \quad \sum_{\mu \in B} \chi^\lambda(\mu) \chi^\alpha(\mu) \chi^\beta(\mu) \geq 0.$$

Proof. Diagonalize $M_B(\lambda)$ by the character table to obtain (7.4). Entrywise nonnegativity follows because $M_B(\lambda)$ is a Kronecker multiplication matrix and powers preserve nonnegative integral entries. $\square$

Again, the nonnegativity of a power of a nonnegative multiplication matrix is elementary. What is retained for the OPAC architecture is the exact class-side identity (7.4), especially the $j = 1$ cancellation of all centralizer weights while preserving the constituent labels α, β . We therefore use (7.4) as a constraint on any future coupling theorem, not as a claim of a new general principle about character tables.

Because every $\mu \in B$ is even,

$$(7.6) \quad H_j(\alpha', \beta; \lambda) = H_j(\alpha, \beta; \lambda) = H_j(\alpha, \beta'; \lambda).$$

Equivalently $JM_B = M_B = M_BJ$, so M_B annihilates the $J = -1$ sector and factors through the quotient identifying α with α' . Its nonzero spectral sector has dimension r_n .

This is substantially stronger than scalar moment domination, but it still concerns only B . The analogous localizer for $N = \text{all non-distinct-odd classes}$ contains both B and the odd sector. The difference that isolates odd classes is not known to be entrywise nonnegative, and this is precisely where a new matrix comparison is required.

Nested PSD selectors alone do not help: they are simultaneously diagonalized by the same character table and therefore merely select nested sets of already-known eigenvectors. Any successful layerwise inertia theorem must use entrywise/module information in addition to spectral support.

8. NO-GO THEOREMS AND ADVERSARIAL CALIBRATION

The route to OPAC-038 is unusually sensitive to plausible but false strengthenings. We record the principal obstructions because they sharply delimit what a valid final theorem must contain.

- N1.** *Componentwise half-negativity is false.* At $n = 5$, $\lambda = (2, 2, 1)$, the Glaisher component $\{(3, 2), (3, 1, 1)\}$ has values $(-1, -1)$. Deficit payment must therefore be nonlocal across components.
- N2.** *Commuting Hall matching is too restrictive.* Britnell–Wildon give class-capacity matchings in cyclic extensions $[2, 3]$, but at $n = 5$, $\lambda = (2, 2, 1)$, the even non-split class $(3, 1, 1)$ has value -1 and every odd class commuting with it also has value -1 . Thus no commuting sign-monotone matching exists.
- N3.** *One-fibre coarsening positivity is false.* The Fourier coefficients in Theorem 5.1 can be negative, and an entire binary coarsening fibre can be unfavorable.
- N4.** *Scalar moments do not determine sign counts.* The example following (6.7) satisfies all scalar moment inequalities but fails the desired negative-count comparison.
- N5.** *Generic split-fusion matrices do not imply inertia.* The 2×2 example after (4.4) satisfies $A, B \geq 0$ and $A - B$ diagonal but violates the hoped-for nonnegative-inertia bound.
- N6.** *Antisymmetric-only Kronecker certificates fail.* For a conjugate pair ν, ν' , the exact quadratic form is

$$(e_\nu - e_{\nu'})^T K_\lambda (e_\nu - e_{\nu'}) = 2(g(\nu, \nu, \lambda) - g(\nu, \nu, \lambda')),$$

so positivity of $g(\nu, \nu, \lambda)$ alone is insufficient. At $\lambda = (4, 4, 3) \vdash 11$, the available zero and positive odd-class spectral dimensions are too small for an antisymmetric-only certificate.

- N7.** *Nested spectral selectors are tautological without module coupling.* Their ranges are simply class-eigenvector spans.

These failures are not peripheral. They explain why the remaining theorem must simultaneously be unweighted, nonlocal, and constituent-sensitive.

9. WHAT REMAINS FOR OPAC-038

The results above do not constitute a reduction of OPAC-038 to a proved finite list of subproblems. They do, however, identify two distinct unresolved couplings that any continuation of the present approach must address.

9.1. Balanced-sector coupling. Let B be the even non-split classes and O the odd classes. Both have cardinality r_n . Sundaram's modules give scalar domination $B_j \geq |O_j|$ for every j , and Theorem 7.2 lifts the B side to a full constituent-level nonnegative matrix hierarchy. The missing theorem is a matrix coupling between B and O through the non-distinct-odd module Φ_N . A successful statement must be strong enough to control negative support or inertia, not merely traces.

A natural target is a factorization or domination on the $J = +1$ quotient that transfers each unit of odd negative excess into unused balanced-even capacity. It cannot be a generic consequence of moment inequalities or simultaneous diagonalization; it must use the Kronecker multiplication structure of $M_B(\lambda)$ and $M_N(\lambda)$.

9.2. Exceptional-sector coupling. The $q(n)$ distinct-odd classes are exactly the classes absent from $B \cup O$ and exactly the split classes of A_n . Theorem 4.2 assigns each negative exceptional value $\chi^\lambda(\delta_i) < 0$ a crossed-channel surplus

$$b_{ii} - a_{ii} = -\chi^\lambda(\delta_i) > 0.$$

The missing theorem must inject or otherwise charge these multiplicity surpluses into balanced capacity left unused after Interface I. This is the representation-theoretic analogue of the Glaisher deficit inequality (1.1).

9.3. What the balanced-hook case demonstrates. The balanced-hook analysis shows what such a payment theorem can look like in a tractable family. Above the one-third scale it is an explicit injection. In the range $k/4 < m \leq k/3$ it becomes a global matching problem involving two-part odd refinements and all-odd target constructions. The infinite family admitting no favorable two-part odd refinement proves that no single local refinement rule can suffice, while the parity-complete all-odd construction proves that local target existence is abundant. The unresolved issue is collision control. This is a concrete model of the same local-to-global obstruction present in the general problem.

Open matching problem for balanced hooks. In the range $k/4 < m \leq k/3$, determine whether the bipartite graph from strict one-even negative-negative sources to the union of favorable canonical odd refinements and parity-complete all-odd targets always has a matching saturating the source side. The theorems above prove sourcewise target existence in broad regimes and exhibit infinite obstructions to using odd refinements alone, but they do not prove this global matching statement.

Programmatic target. A complete proof would require a constituent-level capacity certificate that pays the odd-sector negative excess using B and then absorbs every remaining negative distinct-odd value through the split-fusion surpluses of Theorem 4.2. We do not state this as a formal conjecture: without a specified class of admissible certificates it would be too close to a reformulation of OPAC-038 itself. The point of this discussion is therefore diagnostic rather than theorematic: it separates two independently proved mechanisms and records the coupling information that is still missing.

10. FURTHER EXACT IDENTITIES AT EVEN BINARY SCALES

The terminal odd theory does not extend one edge at a time to even 2-adic scales. The correct algebraic unit is at least a binary square. For a general hook coefficient $X_r(\mu) = H_r^{(|\mu|)}(\mu)$ one has

Lemma 10.1 (One-scale Hadamard descent). *For every partition ρ and positive integer a ,*

$$(10.1) \quad X_r(\rho \cup (a, a)) - X_r(\rho \cup (2a)) = 2(-1)^{a+1} X_{r-a}(\rho \cup a).$$

Proof. Set $s = (-t)^a$ in (3.2). The edge-factor difference is $(1-s)^2 - (1-s^2) = -2s(1-s)$ and coefficient extraction gives the claim. $\square$

Theorem 10.2 (Binary-square descent). *For positive integers a, b ,*

$$(10.2) \quad \begin{aligned} & X_r(\rho \cup (a, a, b, b)) - X_r(\rho \cup (2a, b, b)) \\ & - X_r(\rho \cup (a, a, 2b)) + X_r(\rho \cup (2a, 2b)) \\ & = 4(-1)^{a+b} X_{r-a-b}(\rho \cup a \cup b). \end{aligned}$$

Proof. Apply (10.1) successively in the a and b directions. $\square$

Equation (10.2) is not itself a positivity theorem, but it shows that even-scale cancellation naturally lives on higher-dimensional binary cells. A future global proof may need capacitated identities on such cells rather than edgewise monotonicity.

11. REPRODUCIBILITY AND SCOPE OF THE CLAIMS

The theorem statements in this article are symbolic and their proofs are contained in the text. Exact computation was used during discovery to test formulas and falsify stronger conjectures, but this version does not report finite verification ranges for which the corresponding scripts and raw machine-readable outputs are not deposited with the article. Accordingly, no computational claim is needed to assess the validity of the stated theorems. A future supplementary release may archive independent implementations and exact certificates; such material would be supplementary evidence rather than part of the logical proof.

12. CONCLUSION

The present work changes the shape of OPAC-038 without closing it. Binary Glaisher compression identifies the exact exceptional sector and, for balanced hooks, gives a detailed local payment theory with explicit injections, absorption reservoirs, and infinite obstructions to overly local rules. Alternating-group restriction converts every exceptional principal-hook value into an exact split-fusion multiplicity imbalance and strengthens character nonvanishing to a magnitude inequality. Plethystic coarsening supplies a useful, but largely formal, Fourier language in which established plethystic Murnaghan–Nakayama coefficients give sparse residual spectra and Parseval capacities. More directly relevant to the global problem, conjugation modules and Sundaram’s Schur-positive class sums cross the weighted-to-unweighted barrier and give exact OPAC-specific moment domination. The associated multiplication matrices retain constituent labels and have the exact balanced rank r_n , but their basic nonnegativity and spectral diagonalization are standard and are used here as infrastructure rather than advertised as independent novelty.

The remaining gap is substantial but more structured than the original sign-count formulation. One unresolved task is to couple the balanced even and odd reservoirs at matrix/support level; a second is to convert negative distinct-odd split-fusion surpluses into compatible balanced

capacity. The no-go results show that neither scalar moments, local commuting matchings, generic nonnegative matrices, nor selector nesting can substitute for this coupling. These are research directions suggested by the proved results, not an equivalence theorem or a claimed reduction of OPAC-038. Establishing either coupling in a form that controls negative support would constitute further progress; additional ideas may still be required for a complete solution.

AI ACKNOWLEDGEMENT

This manuscript was generated entirely with OpenAI’s ChatGPT as the research-and-writing system used to develop the mathematical arguments, derivations, exact-checking strategy, exposition, organization, literature synthesis, and LaTeX text. ChatGPT is not listed as an author. The named human author selected the research direction, supplied and reviewed the research materials, requested the analyses and revisions, and assumes responsibility for verifying the mathematical claims, references, disclosure requirements, and suitability of the final manuscript before dissemination or submission.

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