

PARITY CONSTRAINTS FOR KRONECKER COEFFICIENTS FROM ALTERNATING-GROUP SPLITTING

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ABSTRACT. Let $\gamma_1, \dots, \gamma_q \vdash n$ be the self-conjugate partitions, let δ_i be the distinct-odd partition formed by the principal hook lengths of γ_i , and write $\text{Res}_{A_n}^{S_n} \chi^{\gamma_i} = \alpha_i + \beta_i$. For $\lambda \neq \lambda'$, put $\psi_\lambda = \text{Res}_{A_n}^{S_n} \chi^\lambda$. The classical S_n/A_n splitting argument used in the principal-hook criterion of Pak–Panova–Vallejo may be retained at the level of the two A_n constituents rather than recombined immediately. Doing so gives

$$\langle \psi_\lambda \alpha_i, \alpha_j \rangle = \frac{g(\lambda, \gamma_i, \gamma_j) + \chi^\lambda(\delta_i) \delta_{ij}}{2}, \quad \langle \psi_\lambda \alpha_i, \beta_j \rangle = \frac{g(\lambda, \gamma_i, \gamma_j) - \chi^\lambda(\delta_i) \delta_{ij}}{2}.$$

The purpose of this note is to record the quantitative and parity consequences of these identities. In particular, $g(\lambda, \gamma_i, \gamma_i) \geq |\chi^\lambda(\delta_i)|$ with the same parity, whereas $g(\lambda, \gamma_i, \gamma_j)$ is even for $i \neq j$. If $G_\lambda = (g(\lambda, \gamma_i, \gamma_j))$ and $X_\lambda = \text{diag}(\chi^\lambda(\delta_i))$, then $G_\lambda \equiv X_\lambda \pmod{2}$; hence the rank, characteristic polynomial, determinant, and every principal minor of G_λ modulo 2 are determined by the principal-hook values. We also give an exact equality criterion for the diagonal bound and explicit computations in degrees 5 and 8. All Clifford-theoretic ingredients are classical; the contribution is the extraction and organization of these multiplicity and block-parity consequences.

1. INTRODUCTION

For partitions $\lambda, \mu, \nu \vdash n$, the Kronecker coefficient

$$g(\lambda, \mu, \nu) = \langle \chi^\lambda \chi^\mu, \chi^\nu \rangle_{S_n}$$

is the multiplicity of χ^ν in the tensor product $\chi^\lambda \otimes \chi^\mu$. In contrast with Littlewood–Richardson coefficients, no general positive combinatorial rule is known, and even positivity is difficult. Character-theoretic criteria are therefore valuable when they convert a multiplicity question into a tractable evaluation problem.

A particularly useful criterion occurs for self-conjugate partitions. If $\gamma = \gamma'$, the hook lengths on the main diagonal are distinct odd integers summing to n ; their decreasing sequence is a partition $\delta(\gamma)$ of n , the principal-hook partition. The same distinct-odd partitions are precisely the cycle types whose S_n -conjugacy classes split in A_n . Pak, Panova and Vallejo exploited this coincidence in their Main Lemma: if $\chi^\lambda(\delta(\gamma)) \neq 0$, then $g(\lambda, \gamma, \gamma) > 0$ [3]. Their result supplies a powerful nonvanishing certificate for tensor squares.

The A_n splitting mechanism is classical and already enters the proof of the Pak–Panova–Vallejo Main Lemma. Accordingly, we do not present the restriction argument itself as a new representation-theoretic mechanism. The narrower purpose of this note is to retain the two constituent multiplicities separately and to extract consequences that disappear after they are recombined. For self-conjugate γ , the restriction $\chi^\gamma \downarrow_{A_n}$ splits as two irreducibles. Keeping these constituents separate reveals two genuine fusion multiplicities whose sum is a Kronecker coefficient and whose difference is the principal-hook character value. Thus the exceptional value $\chi^\lambda(\delta(\gamma))$ is not merely a test for nonvanishing: it is an exact imbalance between two A_n tensor-product channels.

More precisely, enumerate the self-conjugate partitions of n as γ_i , let $\delta_i = \delta(\gamma_i)$, and write

$$\text{Res}_{A_n}^{S_n} \chi^{\gamma_i} = \alpha_i + \beta_i, \quad \Delta_i = \alpha_i - \beta_i.$$

For $\lambda \neq \lambda'$, the restricted character $\psi_\lambda = \text{Res}_{A_n}^{S_n} \chi^\lambda$ is irreducible and invariant under the outer involution induced by an odd permutation. We first prove

$$\psi_\lambda \Delta_i = \chi^\lambda(\delta_i) \Delta_i.$$

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If

$$a_{ij} = \langle \psi_\lambda \alpha_i, \alpha_j \rangle = \langle \psi_\lambda \beta_i, \beta_j \rangle, \quad b_{ij} = \langle \psi_\lambda \alpha_i, \beta_j \rangle = \langle \psi_\lambda \beta_i, \alpha_j \rangle,$$

then the eigenvector identity gives $a_{ij} - b_{ij} = \chi^\lambda(\delta_i)\delta_{ij}$, while index-two reciprocity gives $a_{ij} + b_{ij} = g(\lambda, \gamma_i, \gamma_j)$. Solving these equations yields the formulas stated in the abstract.

The diagonal consequence

$$g(\lambda, \gamma_i, \gamma_i) \geq |\chi^\lambda(\delta_i)|$$

is the quantitative form of this bookkeeping for non-self-conjugate λ . More importantly for the present note, the same calculation gives an off-diagonal parity theorem and a block congruence modulo 2. The relevant distinction is therefore not a new use of A_n -splitting, but the simultaneous bookkeeping of both split fusion channels. Related refinements of Kronecker squares into symmetric and antisymmetric tensor-square multiplicities have been studied by Bessenrodt and Bowman [4]; those multiplicities are different from the A_n restriction channels considered here. On the class-algebra side, Tekin and Katriel [5] showed that structure constants involving split A_n classes are controlled by character values attached to self-conjugate diagrams and distinct-odd cycle types. Their setting concerns products of class sums rather than the tensor-product multiplicities studied below. The matrix formulation is equally structural. With $A = (a_{ij})$, $B = (b_{ij})$, $G_\lambda = (g(\lambda, \gamma_i, \gamma_j))$ and $X_\lambda = \text{diag}(\chi^\lambda(\delta_i))$, one has $A + B = G_\lambda$ and $A - B = X_\lambda$. Hence the split-fusion block $\begin{pmatrix} A & B \\ B & A \end{pmatrix}$ decomposes into invariant and anti-invariant sectors $G_\lambda \oplus X_\lambda$.

The paper is organized in five sections. Section 2 recalls the required S_n/A_n restriction and reciprocity identities. Section 3 proves the exact split-fusion multiplicities, characterizes equality in the principal-hook bound, and gives complete $n = 5$ and genuinely two-dimensional $n = 8$ computations. Section 4 gives the spectral block decomposition and, rather than treating arbitrary nonnegative blocks as faithful models, separates the necessary arithmetic and representation-ring constraints satisfied by realizable fusion matrices. Section 5 analyzes the self-conjugate multiplying-character case and explains precisely why the two-channel scalar formula is replaced by a four-channel system.

2. PRELIMINARIES AND INDEX-TWO RECIPROCITY

We use standard facts on the representation theory of S_n and A_n ; the relevant Clifford theory is classical and is included only to fix notation and normalizations. The argument below is the same index-two restriction mechanism that underlies the principal-hook criterion in [3]; our emphasis is on retaining the two constituent multiplicities through the calculation.

see [1, 6]. For $\rho \vdash n$, tensoring with the sign character conjugates the partition:

$$\chi^{\rho'} = \chi^\rho \otimes \text{sgn}.$$

Since sgn is trivial on A_n , conjugate S_n -characters have identical restrictions to A_n . Clifford theory gives the familiar dichotomy: if $\rho \neq \rho'$, then $\text{Res}_{A_n}^{S_n} \chi^\rho$ is irreducible; if $\rho = \rho'$, then it is the sum of two distinct irreducibles interchanged by conjugation with any odd permutation.

A class of S_n splits in A_n exactly when its cycle type has distinct odd parts. For self-conjugate γ_i , its principal-hook partition δ_i is such a cycle type. Denote the two A_n classes below it by $C_{\delta_i}^+$ and $C_{\delta_i}^-$. The difference character $\Delta_i = \alpha_i - \beta_i$ is anti-invariant under the outer involution.

Lemma 2.1 (Support and orthogonality of split differences). *The character Δ_i is supported on $C_{\delta_i}^+ \cup C_{\delta_i}^-$ and takes opposite values on the two halves. Moreover*

$$\langle \Delta_i, \Delta_j \rangle_{A_n} = 2\delta_{ij}.$$

Proof. On every non-split A_n class, conjugation by an odd permutation fixes the class and interchanges α_i and β_i , so their values coincide. On the two halves of the split class the same outer involution exchanges the halves, forcing opposite difference values. Orthogonality follows because α_i, β_i are distinct irreducibles and the split pairs arising from different self-conjugate partitions are disjoint. $\square$

Proposition 2.2 (Difference-character eigenvector identity). *If $\lambda \neq \lambda'$, then for every i ,*

$$\psi_\lambda \Delta_i = \chi^\lambda(\delta_i) \Delta_i, \quad \langle \psi_\lambda \Delta_i, \Delta_j \rangle_{A_n} = 2\chi^\lambda(\delta_i) \delta_{ij}.$$

Proof. The two classes in the support of Δ_i fuse to the single S_n class of cycle type δ_i . Hence ψ_λ has the constant value $\chi^\lambda(\delta_i)$ there, while both sides vanish off the support. The inner-product formula follows from Lemma 2.1. $\square$

For completeness we record the index-two identity that supplies the complementary sum relation.

Lemma 2.3 (Index-two reciprocity). *Let $H \triangleleft G$ have index two and let ε be the nontrivial character of G/H , inflated to G . For class functions f, g on G ,*

$$\langle \text{Res}_H^G f, \text{Res}_H^G g \rangle_H = \langle f, g \rangle_G + \langle f, g\varepsilon \rangle_G.$$

Proof. Since $\text{Ind}_H^G \mathbf{1}_H = \mathbf{1}_G + \varepsilon$, Frobenius reciprocity gives

$$\langle \text{Res } f, \text{Res } g \rangle_H = \langle f, \text{Ind Res } g \rangle_G = \langle f, g(\mathbf{1}_G + \varepsilon) \rangle_G.$$

□

Apply Lemma 2.3 with $G = S_n$, $H = A_n$, $\varepsilon = \text{sgn}$, $f = \chi^\lambda \chi^{\gamma_i}$ and $g = \chi^{\gamma_j}$. Since $\gamma_j = \gamma'_j$, one has $\chi^{\gamma_j} \text{sgn} = \chi^{\gamma_j}$, and therefore

$$\langle \psi_\lambda(\alpha_i + \beta_i), \alpha_j + \beta_j \rangle_{A_n} = 2g(\lambda, \gamma_i, \gamma_j).$$

Outer invariance pairs the four summands into two equal pairs. This is the precise index-two mechanism used below.

3. TWO CONSTITUENT MULTIPLICITIES AND PARITY CONSEQUENCES

We now retain the two split constituents rather than recombining them. Define a_{ij} and b_{ij} as in the introduction. They are nonnegative integers, and outer invariance gives the displayed equalities in their definitions.

Theorem 3.1 (Two-channel multiplicity identity). *Let $\lambda \neq \lambda'$ and let γ_i, γ_j be self-conjugate partitions of n . Then*

$$a_{ij} = \frac{g(\lambda, \gamma_i, \gamma_j) + \chi^\lambda(\delta_i)\delta_{ij}}{2}, \quad b_{ij} = \frac{g(\lambda, \gamma_i, \gamma_j) - \chi^\lambda(\delta_i)\delta_{ij}}{2}.$$

Proof. Expanding $\langle \psi_\lambda \Delta_i, \Delta_j \rangle$ and using outer invariance gives

$$2(a_{ij} - b_{ij}) = 2\chi^\lambda(\delta_i)\delta_{ij}.$$

On the other hand, the final identity of Section 2 expands to

$$2(a_{ij} + b_{ij}) = 2g(\lambda, \gamma_i, \gamma_j).$$

Adding and subtracting proves the formulas. □

Corollary 3.2 (Magnitude, parity, and off-diagonal evenness). *For every self-conjugate γ_i ,*

$$g(\lambda, \gamma_i, \gamma_i) \geq |\chi^\lambda(\delta_i)|, \quad g(\lambda, \gamma_i, \gamma_i) \equiv \chi^\lambda(\delta_i) \pmod{2}.$$

If $i \neq j$, then $g(\lambda, \gamma_i, \gamma_j)$ is even and

$$a_{ij} = b_{ij} = \frac{1}{2}g(\lambda, \gamma_i, \gamma_j).$$

Proof. On the diagonal, Theorem 3.1 expresses $(g \pm \chi)/2$ as nonnegative integers. Off the diagonal the character-value correction vanishes. □

The exact formulas also identify the amount by which the principal-hook bound fails to be sharp. This gives a useful equality criterion that is invisible in a mere nonvanishing statement.

Proposition 3.3 (Defect and equality criterion). *For a self-conjugate γ_i , put*

$$x_i = \chi^\lambda(\delta_i), \quad g_i = g(\lambda, \gamma_i, \gamma_i), \quad d_i = \frac{g_i - |x_i|}{2}.$$

Then d_i is a nonnegative integer and

$$d_i = \min\{a_{ii}, b_{ii}\}.$$

Consequently

$$g(\lambda, \gamma_i, \gamma_i) = |\chi^\lambda(\delta_i)|$$

holds if and only if one of the two diagonal split-fusion channels vanishes. More precisely, if $x_i > 0$ equality is equivalent to $b_{ii} = 0$; if $x_i < 0$ it is equivalent to $a_{ii} = 0$; and if $x_i = 0$ equality is equivalent to $a_{ii} = b_{ii} = 0$.

Proof. Theorem 3.1 gives $a_{ii} = (g_i + x_i)/2$ and $b_{ii} = (g_i - x_i)/2$. If $x_i \geq 0$, then $b_{ii} = (g_i - |x_i|)/2$ and $a_{ii} \geq b_{ii}$; if $x_i \leq 0$, the roles are reversed. The asserted equality conditions follow immediately. $\square$

Thus the lower bound has a canonical even defect,

$$g(\lambda, \gamma_i, \gamma_i) - |\chi^\lambda(\delta_i)| = 2d_i,$$

and d_i measures the multiplicity shared by the two competing diagonal channels. Equality is therefore exactly the one-sided-fusion regime.

Explicit examples. Take $n = 5$, $\gamma = (3, 1, 1)$ and $\lambda = (4, 1)$. The partition γ is self-conjugate and its principal-hook partition is $\delta = (5)$. The character $\chi^{(4,1)}$ is the standard character, so at a 5-cycle

$$\chi^{(4,1)}(5) = \# \text{Fix}(5\text{-cycle}) - 1 = -1.$$

For the standard representation one has the classical corner formula

$$g((n-1, 1), \mu, \mu) = c(\mu) - 1,$$

where $c(\mu)$ is the number of removable corners of μ . The diagram $(3, 1, 1)$ has two removable corners, hence

$$g((4, 1), (3, 1, 1), (3, 1, 1)) = 1.$$

Theorem 3.1 therefore gives

$$a = \frac{1 + (-1)}{2} = 0, \quad b = \frac{1 - (-1)}{2} = 1.$$

Thus, in this one-dimensional self-conjugate index set,

$$A = (0), \quad B = (1), \quad G = (1), \quad X = (-1).$$

The full split-fusion matrix on the ordered basis (α, β) is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Passing to the invariant and anti-invariant vectors $\alpha + \beta$ and $\alpha - \beta$ gives eigenvalues 1 and -1 , respectively. Hence the decomposition is exactly

$$G \oplus X = (1) \oplus (-1),$$

and the negative principal-hook value records a one-unit surplus in the crossed channel b .

The first genuinely matrix-valued case already occurs at $n = 8$. There are two self-conjugate partitions,

$$\gamma_1 = (4, 2, 1, 1), \quad \gamma_2 = (3, 3, 2),$$

with principal-hook partitions

$$\delta_1 = (7, 1), \quad \delta_2 = (5, 3).$$

Take $\lambda = (6, 2)$. A direct S_8 character-table computation (equivalently, the Murnaghan–Nakayama rule together with character orthogonality) gives

$$\chi^{(6,2)}(7, 1) = -1, \quad \chi^{(6,2)}(5, 3) = 0,$$

and

$$G_\lambda = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}.$$

Hence

$$X_\lambda = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A = \frac{G_\lambda + X_\lambda}{2} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \frac{G_\lambda - X_\lambda}{2} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}.$$

This example displays both phenomena simultaneously: the off-diagonal Kronecker coefficient is the even integer $g((6, 2), \gamma_1, \gamma_2) = 2$, with $a_{12} = b_{12} = 1$, while the two diagonal principal-hook bounds are strict. Indeed their defects are $d_1 = 2$ and $d_2 = 1$. The corresponding multiplication block is

$$\mathcal{M}_{(6,2)} = \begin{pmatrix} 2 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 \\ 3 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

and the invariant/anti-invariant change of basis gives

$$\mathcal{M}_{(6,2)} \sim G_\lambda \oplus X_\lambda.$$

Since G_λ has eigenvalues 6 and 1, the spectrum of the full operator is $\{6, 1, -1, 0\}$. Thus the $n = 8$ computation is not merely a dimensional illustration: it exhibits a nonzero off-diagonal fusion channel, strict diagonal defects, and the spectral separation of the two sectors in a single example.

4. THE SELF-CONJUGATE KRONECKER BLOCK MODULO TWO

Let the indices range over all self-conjugate partitions of n , and set

$$A = (a_{ij}), \quad B = (b_{ij}), \quad G_\lambda = (g(\lambda, \gamma_i, \gamma_j)), \quad X_\lambda = \text{diag}(\chi^\lambda(\delta_i)).$$

Theorem 3.1 is equivalent to

$$A + B = G_\lambda, \quad A - B = X_\lambda, \quad G_\lambda = 2B + X_\lambda.$$

The restriction of multiplication by ψ_λ to the span of these constituents has matrix

$$\mathcal{M}_\lambda = \begin{pmatrix} A & B \\ B & A \end{pmatrix}.$$

The orthogonal change of coordinates

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ I & -I \end{pmatrix}$$

yields

$$Q^\top \mathcal{M}_\lambda Q = \begin{pmatrix} G_\lambda & 0 \\ 0 & X_\lambda \end{pmatrix}.$$

Thus the invariant sector is the self-conjugate Kronecker block, while the anti-invariant sector is diagonal and has spectrum precisely $\{\chi^\lambda(\delta_i)\}_i$. This simply records the invariant and anti-invariant coordinates of the classical splitting.

The channel formulas also impose a global congruence on the entire Kronecker block, not only on its diagonal entries.

Theorem 4.1 (Parity profile of the self-conjugate Kronecker block). *With the notation above,*

$$G_\lambda \equiv X_\lambda \pmod{2}$$

entrywise. Consequently, over $\mathbf{F}_2$,

$$\text{rank}_{\mathbf{F}_2}(G_\lambda) = \#\{i : \chi^\lambda(\delta_i) \text{ is odd}\},$$

and the characteristic polynomial satisfies

$$\det(tI - G_\lambda) \equiv \prod_i (t - \chi^\lambda(\delta_i)) \pmod{2}.$$

More generally, for every index subset I ,

$$\det G_\lambda[I] \equiv \prod_{i \in I} \chi^\lambda(\delta_i) \pmod{2},$$

where $G_\lambda[I]$ denotes the corresponding principal submatrix. In particular, $\det G_\lambda$ is odd if and only if every principal-hook value $\chi^\lambda(\delta_i)$ is odd.

Proof. The identities $G_\lambda - X_\lambda = 2B$ and $G_\lambda + X_\lambda = 2A$ imply $G_\lambda \equiv X_\lambda \pmod{2}$. Reduction modulo 2 therefore identifies G_λ with a diagonal matrix whose diagonal entries are the reductions of the principal-hook values. The rank and characteristic-polynomial statements follow immediately, and the same argument applied to each principal submatrix gives the minor congruence. $\square$

Thus the parity pattern of all principal-hook values determines the complete reduction of the self-conjugate Kronecker block modulo 2. This packages the diagonal parity congruence and off-diagonal evenness of Corollary 3.2 into a matrix-level invariant.

The congruence theorem is the main matrix-level consequence: it converts the constituent calculation into a statement about the entire self-conjugate Kronecker block. It should not be confused with an inertia

theorem. Indeed, entrywise nonnegativity of the two constituent-multiplicity matrices does not control the inertia of G_λ without additional representation-theoretic input. For example,

$$B = \begin{pmatrix} 4 & 4 \\ 4 & 1 \end{pmatrix}, \quad X = \text{diag}(-1, -1), \quad A = B + X = \begin{pmatrix} 3 & 4 \\ 4 & 0 \end{pmatrix}$$

satisfy the same elementary nonnegativity and diagonal-difference conditions, but $G = A + B = \begin{pmatrix} 7 & 8 \\ 8 & 1 \end{pmatrix}$ has eigenvalues $4 \pm \sqrt{73}$. Thus no stronger spectral conclusion is claimed from these matrix identities alone.

The relation with [3] is deliberately conservative. Their proof already uses the classical splitting of self-conjugate characters on restriction to A_n . We therefore regard the half-sum/half-difference calculation as an unpacking of that mechanism, not as a new Clifford-theoretic principle. What is recorded here are its quantitative consequences: the absolute-value bound, the exact equality defect, off-diagonal evenness, and the block congruence modulo 2. We make no categorical priority claim for the underlying restriction calculation. The tensor-square splitting of Bessenrodt–Bowman [4] and the split-class algebra formulas of Tekin–Katriel [5] are nearby but concern different decompositions. The identification of the anti-invariant eigenvalues with principal-hook character values is special to S_n/A_n .

5. THE SELF-CONJUGATE MULTIPLIER AND FURTHER DIRECTIONS

The hypothesis $\lambda \neq \lambda'$ is exactly what makes the preceding theory two-channel. We now make precise what changes when $\lambda = \lambda'$. Write

$$\text{Res}_{A_n}^{S_n} \chi^\lambda = \rho^+ + \rho^-, \quad (\rho^+)^t = \rho^-$$

for an odd permutation t , and retain $\text{Res} \chi^{\gamma_i} = \alpha_i + \beta_i$. For a fixed sign $\varepsilon \in \{+, -\}$ multiplication by ρ^ε has four natural channels

$$u_{ij}^\varepsilon = \langle \rho^\varepsilon \alpha_i, \alpha_j \rangle, \quad v_{ij}^\varepsilon = \langle \rho^\varepsilon \alpha_i, \beta_j \rangle, \quad w_{ij}^\varepsilon = \langle \rho^\varepsilon \beta_i, \alpha_j \rangle, \quad z_{ij}^\varepsilon = \langle \rho^\varepsilon \beta_i, \beta_j \rangle.$$

Conjugation by t does not preserve a fixed multiplier: it exchanges ρ^+ and ρ^- . Hence

$$u_{ij}^+ = z_{ij}^-, \quad v_{ij}^+ = w_{ij}^-, \quad w_{ij}^+ = v_{ij}^-, \quad z_{ij}^+ = u_{ij}^-.$$

The ordinary Kronecker coefficient controls only the total mass of these four channels.

Proposition 5.1 (Four-channel mass identity). *If $\lambda, \gamma_i, \gamma_j$ are self-conjugate, then*

$$u_{ij}^+ + v_{ij}^+ + w_{ij}^+ + z_{ij}^+ = g(\lambda, \gamma_i, \gamma_j),$$

and the same identity holds with $+$ replaced by $-$.

Proof. Index-two reciprocity applied to $f = \chi^\lambda \chi^{\gamma_i}$ and $g = \chi^{\gamma_j}$ gives

$$\langle (\rho^+ + \rho^-)(\alpha_i + \beta_i), \alpha_j + \beta_j \rangle_{A_n} = 2g(\lambda, \gamma_i, \gamma_j).$$

Conjugation by an odd permutation exchanges the complete $+$ and $-$ contributions, so they are equal. Each must therefore equal $g(\lambda, \gamma_i, \gamma_j)$, which is precisely the asserted four-term sum. $\square$

This proposition explains why Theorem 3.1 does not extend by simply replacing ψ_λ with one of $\rho^\pm$. In the non-self-conjugate case the multiplier is outer-invariant, forcing the four matrix entries into two paired channels and allowing the difference-character eigenvalue to determine their difference. For a self-conjugate multiplier, the outer involution exchanges the two multiplication operators themselves. A second scalar identity would require the actual split values of $\rho^\pm$ on the relevant classes. Classical formulas show that these values contain square-root terms on the split class corresponding to λ ; see, for example, the alternating-group character formulas in [1]. Thus the natural extension is genuinely a coupled four-channel problem rather than a missing special case of the two-channel theorem.

The calculation above is best viewed as a quantitative extraction from the classical S_n/A_n splitting argument. Principal hooks identify the relevant split classes and make the anti-invariant eigenvalues ordinary S_n character values. Within this specialization, the fusion-defect formula characterizes sharpness of the principal-hook bound, while $G_\lambda \equiv X_\lambda \pmod{2}$ determines the complete self-conjugate Kronecker block modulo 2.

Two problems remain natural. First, one may ask for an intrinsic characterization of those parity-compatible pairs (G, X) that are realizable by A_n fusion, incorporating the associativity constraints of $R(A_n)$. Second, in the self-conjugate multiplier case, the four-channel mass identity above should be

combined with the explicit split-character values to determine which additional linear combinations of u, v, w, z are forced by the principal-hook geometry. These questions are representation-theoretic rather than purely matrix-theoretic and delineate the point at which the elementary index-two argument gives way to finer structure.

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