INTEGRAL LATTICE AND PARSEVAL CONSEQUENCES OF THE PLETHYSTIC MURNAGHAN–NAKAYAMA RULE

SIMON WATTS

ABSTRACT. Fix $\lambda \vdash n$, integers $k, m \geq 1$, and a residual partition τ with $|\tau| + km = n$. For $\nu \vdash m$ set $F_\tau(\nu) = \chi^\lambda(\tau \cup k\nu)$. We record a Fourier formulation of this complete equal-part coarsening profile and then combine it with the plethystic Murnaghan–Nakayama rule. The profile is an integral virtual character of S_m, with coefficient $\langle s_\lambda, p_\tau s_\alpha[p_k]\rangle$ at χ^α. In the pure-fibre case the trivial and sign coefficients are therefore in $\{0, \pm 1\}$. More significantly, after the residual cycle type is allowed to vary, the trivial coarsening mode is itself an S_r-virtual character whose irreducible coefficients are all $0, \pm 1$. This gives an exact Parseval identity: its class-measure energy equals the number of nonzero plethystic Murnaghan–Nakayama coefficients. We also give the corresponding cross-Parseval identity, coefficient-recovery formula, and an even/odd coarsening decomposition related by partition conjugation. The plethystic rule itself is prior work. Beyond the two-stage transform and its exact energy consequences, we prove that the signed residual coefficient vectors generate the full integral character lattice: the coefficient matrix of multiplication by $h_m[p_k]$ has full column rank and Smith invariants all equal to one. Thus every irreducible character of the residual symmetric group is an integral combination of residual coarsening profiles. We conclude with a unimodular-basis conjecture, verified computationally in small degrees.

1. Introduction

The classical Murnaghan–Nakayama rule connects symmetric-group character values with signed border-strip combinatorics. Its plethystic analogue describes the Schur expansion of

$$(p_k \circ h_m)s_\mu = h_m[p_k]s_\mu$$

by a signed k-decomposability condition. Combinatorial, abacus, and vertex-operator proofs are available; see Wildon [6], Turek [5], and Cao–Jing–Liu [1]. In particular, the relevant Schur coefficient is either 0 or a single sign. We take this theorem as an input and do not claim a new proof or strengthening of the plethystic Murnaghan–Nakayama rule.

Our purpose is narrower. Fix $\lambda \vdash n$, integers $k, m \geq 1$, and $r = n - km \geq 0$. For $\tau \vdash r$ and $\nu \vdash m$ define

$$F_\tau(\nu) = \chi^\lambda(\tau \cup k\nu), \qquad k\nu = (k\nu_1, k\nu_2, \ldots).$$

The partitions ν parameterize all coarsenings of m equal k-cycles. We ask what the complete class function F_τ looks like when expanded in the irreducible character basis of S_m, and what remains after one averages over this coarsening variable.

The first answer is an exact Fourier identity:

$$F_\tau(\nu) = \sum_{\alpha \vdash m} c_\alpha(\tau)\chi^\alpha(\nu), \qquad c_\alpha(\tau) = \langle s_\lambda, p_\tau s_\alpha[p_k]\rangle \in \mathbb{Z}.$$

This is a formal consequence of Frobenius characteristic and character orthogonality. Its usefulness is that the extremal coefficients $\alpha = (m), (1^m)$ can be fed directly into the plethystic Murnaghan–Nakayama rule.

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The less immediate point appears when the residual partition τ itself varies. Define

$$C_+^\lambda(\tau) = \sum_{\nu \vdash m} \frac{\chi^\lambda(\tau \cup k\nu)}{z_\nu}.$$

Then, as a class function on S_r,

$$C_+^\lambda = \sum_{\mu \vdash r} \varepsilon_\mu^\lambda \chi^\mu, \qquad \varepsilon_\mu^\lambda = \langle s_\lambda, s_\mu h_m[p_k] \rangle \in \{0, \pm 1\}.$$

Thus a whole residual family of averaged character values is controlled by a sparse signed spectrum. Parseval gives

$$\sum_{\tau \vdash r} \frac{|C_+^\lambda(\tau)|^2}{z_\tau} = \#\{\mu \vdash r : \varepsilon_\mu^\lambda \neq 0\}.$$

We also record the bilinear version, which identifies correlations of two residual profiles with the signed overlap of their plethystic supports. These identities are elementary once the two transforms are placed side by side, but they do not appear as the usual statement of the plethystic Murnaghan–Nakayama rule.

The paper is framed around consequences that are not part of the usual statement of the plethystic rule. Section 2 establishes the coarsening transform. Section 3 treats the pure-fibre extremal modes and a small example. Section 4 proves the residual transform, Parseval and cross-Parseval identities, and the parity-resolved two-mode formulas. Section 5 proves the integral spanning and Smith-normal-form theorem and formulates a stronger unimodular-basis conjecture.

Relation with the existing plethystic rule. The distinction between the input theorem and the deductions made below is important. Wildon's formulation expresses $s_\mu(p_k \circ h_m)$ as a signed sum of Schur functions indexed by k-decomposable skew shapes; Turek gives a labelled-abacus proof of the same rule; and Cao–Jing–Liu recover the rule from vertex operators and a determinant formula. Thus the signed-unit Schur coefficient is not new here. What is less visible in those formulations is that, after one fixes the outer partition λ and regards the coarsening type ν as a variable, ordinary character orthogonality produces a second representation-theoretic object. The point of the present paper is to isolate that object and to keep both variables—the coarsening variable and the residual cycle variable—long enough for Parseval and polarization to become exact combinatorial identities.

This viewpoint also explains why no positivity assertion should be expected for the complete profile. The Fourier coefficients are integral because Adams operations preserve the integral Schur lattice, but they can be negative. The signed-unit phenomenon enters only in the two extremal coarsening modes and, after the residual variable is transformed, in the residual spectrum. Separating these two facts prevents a formal Fourier inversion from being mistaken for a new plethystic positivity theorem.

2. The coarsening character transform

We use the Hall inner product normalized by

$$\langle p_\rho, p_\sigma \rangle = z_\rho \delta_{\rho\sigma}, \qquad s_\lambda = \sum_{\rho \vdash |\lambda|} \frac{\chi^\lambda(\rho)}{z_\rho} p_\rho,$$

where $z_\rho = \prod_i i^{m_i(\rho)} m_i(\rho)!$. Plethysm is written with brackets. Since

$$s_\alpha[p_k] = \sum_{\rho \vdash m} \frac{\chi^\alpha(\rho)}{z_\rho} p_{k\rho},$$

we obtain the following.

Theorem 2.1 (Coarsening transform)**.** *For every $\tau \vdash r = n - km$, the class function F_τ on S_m has the expansion*

$$F_\tau(\nu) = \sum_{\alpha \vdash m} c_\alpha(\tau)\chi^\alpha(\nu),$$

where

$$c_\alpha(\tau) = \langle s_\lambda, p_\tau s_\alpha[p_k]\rangle = \sum_{\rho \vdash m} \frac{\chi^\alpha(\rho)F_\tau(\rho)}{z_\rho} \in \mathbb{Z}.$$

Hence F_τ is an integral virtual character of S_m.

Proof. Expanding $s_\alpha[p_k]$ gives

$$\langle s_\lambda, p_\tau s_\alpha[p_k]\rangle = \sum_{\rho \vdash m} \frac{\chi^\alpha(\rho)}{z_\rho}\langle s_\lambda, p_{\tau \cup k\rho}\rangle = \sum_{\rho \vdash m} \frac{\chi^\alpha(\rho)\chi^\lambda(\tau \cup k\rho)}{z_\rho}.$$

This is the irreducible Fourier coefficient of F_τ. Orthogonality gives the expansion. Integrality follows because $p_\tau s_\alpha[p_k]$ has integral Schur coefficients: multiplication by each power sum is integral by the ordinary Murnaghan–Nakayama rule, and the Adams operation $f \mapsto f[p_k]$ preserves the integral Schur lattice. □

For the trivial and sign characters of S_m write

$$c_+(\tau) = \sum_{\nu \vdash m} \frac{\chi^\lambda(\tau \cup k\nu)}{z_\nu}, \qquad c_-(\tau) = \sum_{\nu \vdash m} \frac{\mathrm{sgn}(\nu)\chi^\lambda(\tau \cup k\nu)}{z_\nu}.$$

The involution ω satisfies $\omega(s_\lambda) = s_{\lambda'}$ and $\omega(p_j) = (-1)^{j-1}p_j$. In particular,

$$\omega(h_m[p_k]) = (-1)^{m(k-1)}e_m[p_k].$$

2.1. **Fourier inversion and uniqueness.** The character table gives not only the forward coefficients but also an exact inverse description. If

$$F_\tau = \sum_{\alpha \vdash m} c_\alpha(\tau)\chi^\alpha,$$

then the vector $(c_\alpha(\tau))_{\alpha \vdash m}$ is uniquely determined by the complete coarsening fibre. Conversely the values at every merged cycle type are recovered by the same character table. In particular, any proposed identity among complete coarsening profiles can be checked coefficientwise in the representation ring of S_m. This elementary observation is useful because it turns a family of character values of S_n into a finite integral vector indexed by irreducibles of the smaller group S_m.

Proposition 2.2 (Integral lattice statement)**.** *For fixed λ, k, m and residual τ, the vector*

$$\left(\chi^\lambda(\tau \cup k\nu)\right)_{\nu \vdash m}$$

lies in the integral character lattice of S_m. Equivalently, for every generalized character $\theta = \sum_\alpha u_\alpha \chi^\alpha$ of S_m with $u_\alpha \in \mathbb{Z}$,

$$\sum_{\nu \vdash m} \frac{\theta(\nu)\chi^\lambda(\tau \cup k\nu)}{z_\nu} = \sum_{\alpha \vdash m} u_\alpha c_\alpha(\tau) \in \mathbb{Z}.$$

Proof. The first assertion is Theorem 2.1. Pairing its character expansion with θ proves the displayed identity. □

3. Pure-fibre extremal modes

Assume $r = 0$. Then

$$c_+(\lambda; k, m) = \langle s_\lambda, h_m[p_k]\rangle, \qquad c_-(\lambda; k, m) = \langle s_\lambda, e_m[p_k]\rangle.$$

Theorem 3.1 (Extremal modes)**.** *For every $\lambda \vdash km$,*

$$c_+(\lambda; k, m) \in \{0, \pm 1\}, \qquad c_-(\lambda; k, m) = (-1)^{m(k-1)} c_+(\lambda'; k, m).$$

In particular $c_-(\lambda; k, m) \in \{0, \pm 1\}$.

Proof. The first assertion is the $\mu = \varnothing$ case of the plethystic Murnaghan–Nakayama rule. Applying the isometric involution ω to $h_m[p_k]$ gives the second identity. ☐

If

$$E = \sum_{\substack{\nu \vdash m \\ \mathrm{sgn}(\nu)=1}} \frac{\chi^\lambda(k\nu)}{z_\nu}, \qquad O = \sum_{\substack{\nu \vdash m \\ \mathrm{sgn}(\nu)=-1}} \frac{\chi^\lambda(k\nu)}{z_\nu},$$

then $E = (c_+ + c_-)/2$ and $O = (c_+ - c_-)/2$. Hence $E, O \in \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$. This is a cancellation statement for class-measure-weighted sums, not a pointwise positivity statement.

An explicit example. Take $k = m = 2$ and $\lambda = (3, 1) \vdash 4$. The partitions of 2 are $(1, 1)$ and (2), giving cycle types $(2, 2)$ and (4) in S_4. Since $\chi^{(3,1)}$ is the standard character,

$$F(1, 1) = \chi^{(3,1)}(2, 2) = -1, \qquad F(2) = \chi^{(3,1)}(4) = -1.$$

As $z_{(1,1)} = z_{(2)} = 2$,

$$c_{(2)} = -1, \qquad c_{(1,1)} = 0,$$

so $F = -\chi^{(2)}$. This shows concretely why "virtual" cannot be replaced by "genuine" in Theorem 2.1.

3.1. What the pure-fibre theorem does and does not say.

The extremal coefficients c_+ and c_- are averages with respect to the trivial and sign characters of S_m. Their three-valued nature is therefore a global cancellation statement over an entire coarsening fibre. It does not assert that individual values $\chi^\lambda(k\nu)$ have controlled sign or magnitude. This distinction matters in applications: a bounded Fourier mode can coexist with large pointwise character values. The residual theory below retains precisely enough information to measure this cancellation in the natural class-function norm.

4. Residual spectra, Parseval, and parity resolution

Let $r = n - km$ be arbitrary and define

$$C_+^\lambda(\tau) = \sum_{\nu \vdash m} \frac{\chi^\lambda(\tau \cup k\nu)}{z_\nu} \qquad (\tau \vdash r).$$

Theorem 4.1 (Residual signed spectrum)**.** *As a class function of S_r,*

$$C_+^\lambda(\tau) = \sum_{\mu \vdash r} \varepsilon_\mu^\lambda \chi^\mu(\tau), \qquad \varepsilon_\mu^\lambda = \langle s_\lambda, s_\mu h_m[p_k]\rangle \in \{0, \pm 1\}.$$

Equivalently, every coefficient can be recovered directly from the residual profile by

$$\varepsilon_\mu^\lambda = \sum_{\tau \vdash r} \frac{C_+^\lambda(\tau)\chi^\mu(\tau)}{z_\tau}.$$

Proof. The μ-Fourier coefficient of C_+^λ is

$$\sum_{\tau\vdash r}\sum_{\nu\vdash m}\frac{\chi^\lambda(\tau\cup k\nu)\chi^\mu(\tau)}{z_\tau z_\nu}=\langle s_\lambda, s_\mu h_m[p_k]\rangle.$$

The plethystic Murnaghan–Nakayama rule makes this coefficient 0 or a single sign. Fourier inversion gives the stated expansion and recovery formula. ☐

Set

$$A(\lambda;k,m)=\#\{\mu\vdash r:\varepsilon_\mu^\lambda\neq 0\}.$$

Corollary 4.2 (Exact Parseval capacity)**.** *One has*

$$\sum_{\tau\vdash r}\frac{|C_+^\lambda(\tau)|^2}{z_\tau}=A(\lambda;k,m).$$

Consequently $|C_+^\lambda(\tau)|\le\sqrt{A(\lambda;k,m)z_\tau}$ *for each* τ*.*

Proof. Apply character orthonormality to Theorem 4.1; the pointwise estimate follows from the nonnegative summands in the norm identity. ☐

The same argument has a useful bilinear form.

Proposition 4.3 (Cross-Parseval identity)**.** *Let* $\lambda,\kappa\vdash n$ *and use the same* k,m,r*. Then*

$$\sum_{\tau\vdash r}\frac{C_+^\lambda(\tau)C_+^\kappa(\tau)}{z_\tau}=\sum_{\mu\vdash r}\varepsilon_\mu^\lambda\varepsilon_\mu^\kappa.$$

Thus the class-measure correlation of two residual profiles is exactly the signed overlap of their plethystic Murnaghan–Nakayama supports, and in particular

$$\left|\sum_{\tau\vdash r}\frac{C_+^\lambda(\tau)C_+^\kappa(\tau)}{z_\tau}\right|\le\sqrt{A(\lambda;k,m)A(\kappa;k,m)}.$$

Proof. Expand both profiles in the orthonormal irreducible basis and use Parseval; the inequality is Cauchy–Schwarz on the coefficient vectors. ☐

Now define the sign-weighted mode

$$C_-^\lambda(\tau)=\sum_{\nu\vdash m}\frac{\operatorname{sgn}(\nu)\chi^\lambda(\tau\cup k\nu)}{z_\nu}.$$

Since $\chi^{\lambda'}(\rho)=\operatorname{sgn}(\rho)\chi^\lambda(\rho)$ and $\operatorname{sgn}(k\nu)=(-1)^{m(k-1)}\operatorname{sgn}(\nu)$,

$$C_-^\lambda(\tau)=(-1)^{m(k-1)}\operatorname{sgn}(\tau)C_+^{\lambda'}(\tau).$$

Define

$$E_\tau=\frac{C_+^\lambda(\tau)+C_-^\lambda(\tau)}{2},\qquad O_\tau=\frac{C_+^\lambda(\tau)-C_-^\lambda(\tau)}{2}.$$

With $\|f\|^2=\sum_{\tau\vdash r}|f(\tau)|^2/z_\tau$, the parallelogram identity gives

$$\|E\|^2+\|O\|^2=\frac{A(\lambda;k,m)+A(\lambda';k,m)}{2},$$

and polarization gives

$$\|E\|^2-\|O\|^2=\langle C_+^\lambda, C_-^\lambda\rangle.$$

These are exact identities; no positivity hypothesis on the character values is used.

4.1. **Support geometry and pairwise correlations.** The cross-Parseval identity has a useful geometric interpretation. Associate to λ the signed support vector

$$e^{\lambda} = (\varepsilon_{\mu}^{\lambda})_{\mu \vdash r} \in \{0, \pm 1\}^{p(r)}.$$

Then C_{+}^{λ} is the image of e^{λ} under the unitary character-table transform for class measure, and

$$\langle C_{+}^{\lambda}, C_{+}^{\kappa} \rangle = \langle e^{\lambda}, e^{\kappa} \rangle.$$

Thus orthogonality of two residual profiles is equivalent to cancellation in the signed overlap of their admissible plethystic border-strip supports. Equality in the Cauchy–Schwarz bound occurs exactly when the two nonzero signed support vectors are proportional; since their entries lie in $\{0, \pm 1\}$, this means that their supports coincide and their signs agree everywhere or disagree everywhere. This gives a sharp equality criterion that is not visible from a pointwise estimate on character values.

Corollary 4.4 (Equality in cross-capacity)**.** *Assume $A(\lambda; k, m)A(\kappa; k, m) > 0$. Then*

$$|\langle C_{+}^{\lambda}, C_{+}^{\kappa} \rangle| = \sqrt{A(\lambda; k, m)A(\kappa; k, m)}$$

if and only if $A(\lambda; k, m) = A(\kappa; k, m)$ and there is $\eta \in \{\pm 1\}$ such that $\varepsilon_{\mu}^{\lambda} = \eta \varepsilon_{\mu}^{\kappa}$ for every $\mu \vdash r$.

Proof. Equality in Cauchy–Schwarz makes the two real support vectors proportional. Their nonzero coordinates all have absolute value one, forcing the proportionality constant to be ± 1 and the supports to coincide. $\square$

4.2. **A finite-dimensional Gram formulation.** For any finite family $\lambda^{(1)}, \ldots, \lambda^{(N)}$ of partitions of $r + km$, the matrix

$$\mathcal{G}_{ab} = \sum_{\tau \vdash r} \frac{C_{+}^{\lambda^{(a)}}(\tau) C_{+}^{\lambda^{(b)}}(\tau)}{z_{\tau}}$$

is positive semidefinite and integral. Indeed, by cross-Parseval it is the Gram matrix of the integer vectors $e^{\lambda^{(a)}}$. Its diagonal entries are the exact capacities $A(\lambda^{(a)}; k, m)$, while its off-diagonal entries record signed support intersections. Consequently every principal minor is a nonnegative integer. This packages all pairwise residual identities into one basis-independent constraint and supplies immediate determinant inequalities beyond the two-profile Cauchy–Schwarz estimate.

4.3. **A residual example and a normalization check.** Take $k = 1$, $m = 2$, $r = 1$, and $\lambda = (2, 1) \vdash 3$. Although $k = 1$ specializes the plethystic rule to the ordinary Pieri/Murnaghan–Nakayama setting, it is the smallest example in which the two transforms and the residual variable can all be seen without suppressing any normalization. There is a unique residual partition $\tau = (1)$ and two coarsening types $\nu = (1, 1), (2)$. The S_3 character $\chi^{(2,1)}$ takes the values

$$\chi^{(2,1)}(1^3) = 2, \qquad \chi^{(2,1)}(2, 1) = 0.$$

Since $z_{(1,1)} = z_{(2)} = 2$, the two extremal coarsening modes are

$$C_{+}^{(2,1)}((1)) = \frac{2}{2} + \frac{0}{2} = 1, \qquad C_{-}^{(2,1)}((1)) = \frac{2}{2} - \frac{0}{2} = 1.$$

On the residual side S_1 has only the trivial character. Pieri's rule gives

$$\varepsilon_{(1)}^{(2,1)} = \langle s_{(2,1)}, s_{(1)} h_2 \rangle = 1,$$

so the residual expansion is $C_{+} = \chi^{(1)}$. Parseval reads simply

$$\sum_{\tau \vdash 1} \frac{|C_{+}(\tau)|^2}{z_{\tau}} = 1 = A((2, 1); 1, 2).$$

The example is intentionally elementary: its purpose is to expose the $1/z_\nu$ and $1/z_\tau$ normalizations and to show that the residual signed spectrum is a second character transform, not an additional convention hidden in the notation.

5. Integral spanning, Smith normal form, and a conjectural basis

The preceding Parseval identities take place over the complex character space. The plethystic Murnaghan–Nakayama rule contains a stronger integral statement that appears after all outer partitions are considered simultaneously. We now record it.

Fix $k, m \geq 1$ and $r \geq 0$, put $n = r + km$, and let

$$E = E_{r;k,m} = \left(\varepsilon_\mu^\lambda\right)_{\lambda \vdash n,\, \mu \vdash r}$$

be the rectangular matrix from Theorem 4.1. Thus the μth column of E is the Schur-coordinate vector of

$$s_\mu h_m[p_k] \in \Lambda_n,$$

and every entry of E lies in $\{0, \pm 1\}$.

Theorem 5.1 (Integral residual spanning and Smith form)**.** *The matrix $E_{r;k,m}$ has rank $p(r)$ over $\mathbb{Q}$, and its image in $\mathbb{Z}^{p(n)}$ is a saturated sublattice. Equivalently, every nonzero Smith invariant of E equals 1:*

$$\mathrm{SNF}(E) = \begin{pmatrix} I_{p(r)} \\ 0 \end{pmatrix}$$

up to unimodular row and column operations. Consequently the signed support vectors

$$e^\lambda = (\varepsilon_\mu^\lambda)_{\mu \vdash r} \in \{0, \pm 1\}^{p(r)}, \qquad \lambda \vdash n,$$

generate the full lattice $\mathbb{Z}^{p(r)}$.

Proof. Let $f = h_m[p_k] \in \Lambda_{km}$. Multiplication by f gives a homomorphism

$$M_f : \Lambda_r \longrightarrow \Lambda_n, \qquad g \longmapsto gf,$$

whose matrix in the Schur bases is E. Since the ring of symmetric functions over $\mathbb{Z}$ is an integral domain and $f \neq 0$, the map is injective; hence E has column rank $p(r)$.

It remains to prove saturation. By the plethystic Murnaghan–Nakayama rule, the Schur expansion of f has coefficients in $\{0, \pm 1\}$ and at least one coefficient is nonzero. Thus f is primitive in the integral polynomial ring $\Lambda_\mathbb{Z} \cong \mathbb{Z}[h_1, h_2, \ldots]$. Suppose that a prime q and $g \in \Lambda_\mathbb{Z}$ satisfy $qg = fh$. Reducing modulo q gives $\bar{f}\,\bar{h} = 0$ in the domain $\Lambda_{\mathbb{F}_q}$. Primitivity gives $\bar{f} \neq 0$, so $\bar{h} = 0$; hence $h = qh_1$ and therefore $g = fh_1$. Iterating over the prime factors of an arbitrary integer shows that $\Lambda_\mathbb{Z}/(f)$ is torsion-free. Taking the homogeneous degree-n component proves that $\operatorname{coker} M_f$ is torsion-free. This is exactly the assertion that all nonzero Smith invariants of E are one.

Finally, the row lattice of an integer matrix with Smith invariants all equal to one is the whole $\mathbb{Z}^{p(r)}$, proving the last assertion. □

Theorem 5.1 has a direct character-theoretic interpretation. The character table is unimodular only after the usual class-measure normalization, so rational spanning of the residual profiles would be automatic from injectivity. The theorem says more: no denominator is required on the irreducible side.

Corollary 5.2 (Integral reconstruction of residual irreducibles)**.** *For every $\rho \vdash r$ there exist integers a_λ, only finitely many nonzero, such that*

$$\chi^\rho(\tau) = \sum_{\lambda \vdash r+km} a_\lambda C_+^\lambda(\tau) \qquad \textit{for every } \tau \vdash r.$$

Thus the residual coarsening profiles span the full generalized-character lattice of S_r, not merely its rational or real vector space.

Proof. By Theorem 5.1, the standard basis vector indexed by ρ is an integral combination of the rows e^λ. Apply the irreducible-character transform of Theorem 4.1. $\square$

5.1. **Signed Hamming geometry.** Because the rows of E lie in $\{0, \pm 1\}^{p(r)}$, the class-measure metric on residual profiles has an exact discrete description. If S_λ and S_κ are the supports of e^λ and e^κ, and $N_{\mathrm{opp}}(\lambda, \kappa)$ is the number of common coordinates on which their signs are opposite, then

$$\|C_+^\lambda - C_+^\kappa\|^2 = |S_\lambda \triangle S_\kappa| + 4N_{\mathrm{opp}}(\lambda, \kappa).$$

Thus the residual transform is an isometry from a signed Hamming-type lattice to the class-function space equipped with class measure. In particular, two residual profiles coincide exactly when their signed plethystic supports coincide.

5.2. **A unimodular-basis conjecture.** Saturation does not, for a general integer matrix, imply that some maximal square minor is unimodular. The special $0, \pm 1$ matrices arising here appear to satisfy this stronger property.

Conjecture 5.3 (Unimodular residual basis)**.** *For every $r \geq 0$ and $k, m \geq 1$, with $n = r + km$, there exist $p(r)$ partitions $\lambda^{(1)}, \ldots, \lambda^{(p(r))} \vdash n$ such that*

$$\det\big(\varepsilon_\mu^{\lambda^{(a)}}\big)_{1 \leq a \leq p(r),\, \mu \vdash r} = \pm 1.$$

Equivalently, one can choose $p(r)$ residual profiles that already form an integral basis of the generalized-character lattice of S_r.

Exact Murnaghan–Nakayama computation verifies Conjecture 5.3 for all cases with $n \leq 10$, $2 \leq k \leq 3$, and $r \geq 1$ (and all admissible m). Theorem 5.1 proves that the gcd of the maximal minors is one; the conjecture asks for the stronger assertion that one maximal minor itself realizes this gcd.

5.3. **Scope and novelty boundary.** The coarsening transform in Theorem 2.1 is Fourier inversion, and the signed-unit coefficients in Theorems 3.1 and 4.1 come from the established plethystic Murnaghan–Nakayama rule [6, 5, 1]. The Parseval and Gram identities are then exact consequences of composing these transforms. Theorem 5.1 is of a different nature: it uses the signed-unit plethystic expansion together with the integral-domain and primitive-element structure of $\Lambda_{\mathbb{Z}}$ to obtain an integral lattice statement and a Smith-normal-form conclusion.

A targeted literature search through the plethystic Murnaghan–Nakayama literature located the iterated plethystic rule but did not locate this residual Smith-normal-form formulation. This is not a claim of exhaustive priority. The theorem is stated here because it gives a falsifiable structural assertion substantially stronger than Parseval alone.

The weighted identities still do not imply unweighted sign-counting bounds, since the factors z_τ vary sharply. Nor does Theorem 5.1 identify a canonical integral basis. Conjecture 5.3 isolates that next discrete question.

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