

EXTREMAL NEGATIVITY IN SYMMETRIC-GROUP CHARACTER TABLES: SOLVED SECTORS AND AN ARITHMETIC SQUARE-CORE REDUCTION

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ABSTRACT. For an irreducible character χ^λ of S_n , let

$$N_n^-(\lambda) = \#\{\mu \vdash n : \chi^\lambda(\mu) < 0\},$$

with one entry counted for each conjugacy class. OPAC-038 asks whether the sign representation eventually maximizes N_n^- . We show that the sign row has exactly $(p(n) - q(n))/2$ negative entries and reduce fixed-degree extremality to the parity-balance inequality

$$E^-(\lambda) \leq O^-(\lambda) + O^0(\lambda).$$

We prove this inequality for every self-conjugate row in every degree. For two-row characters we obtain a staircase-core positivity criterion and a uniform solved range $k < (\sqrt{6}/(4\pi) - \varepsilon)\sqrt{n} \log n$. The pairs $(n-1, 1), (2, 1^{n-2})$ and $(n-2, 2), (2, 2, 1^{n-4})$ are settled in every degree. A uniform character-polynomial argument further shows that, for every $c < 1/2$, OPAC holds eventually whenever either $n - \lambda_1$ or $n - \lambda'_1$ is at most $c \log n / \log \log n$.

For the remaining non-self-conjugate rows we isolate an arithmetic square core. After removing the $q(n)$ distinct-odd split columns and the $q(n)$ self-conjugate rows, the normalized nonsplit-even block is square and invertible, as is the odd block. The split-even columns form a $q(n)$ -dimensional graph correction. After Frobenius scaling this correction is integral and satisfies a determinant-one Gram identity. Both square cores are invertible over every $\mathbb{F}_\ell$ with $\ell > n$, the canonical odd-to-nonsplit-even bridge is an isometry outside at most $q(n)$ directions, and the two cores admit an orthogonal phase normal form together with a canonical mod-2 congruence. An explicit real square-core countermodel proves that these real metric identities alone cannot force the required sign inequality. Finally, the parity deficit is exactly an integral pairing of a virtual character with the conjugation character. Exact computation verifies OPAC through $n = 29$. The global conjecture remains open.

1. INTRODUCTION

Let $\lambda, \mu \vdash n$, and write $\chi^\lambda(\mu)$ for the ordinary irreducible character value of S_n . Conjugacy classes are counted uniformly, not with class-size weights. Hopkins posed the following problem as OPAC-038 [5].

Conjecture 1.1 (OPAC-038). *For all sufficiently large n and every $\lambda \vdash n$,*

$$N_n^-(\lambda) \leq N_n^-((1^n)).$$

The unweighted statistic is not directly controlled by character orthogonality. The first reduction is exact: conjugating partitions turns the problem into a comparison between negative even classes and nonpositive odd classes. Thus zeros are part of the target rather than an error term.

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The main general result is the square-core reduction. Let $\mathcal{D}_n$ be the distinct-odd partitions and $\mathcal{N}_n = E_n \setminus \mathcal{D}_n$. Then

$$|\mathcal{N}_n| = |O_n| = \frac{p(n) - q(n)}{2}.$$

After pairing each non-self-conjugate irreducible with its conjugate and taking symmetric/antisymmetric normalized row combinations, the character table separates into an even block and an odd square block. The self-conjugate/distinct-odd block is unimodular, so complementary orthogonal minors produce a square invertible nonsplit-even block. The split-even columns are a graph over this square core. This converts OPAC for a conjugate pair into the coordinate inequality in Corollary 7.3.

The same reduction has arithmetic structure. Its graph correction is integral after Frobenius scaling, the associated Frobenius Gram form has determinant one, both square cores remain invertible in cross characteristic $\ell > n$, and Miller's cycle-merging congruence relates the nonsplit-even and odd cores modulo 2 [9]. The phase normal form isolates the remaining problem as arithmetic sign rigidity for the actual symmetric-group phase. We also record the explicit real-metric no-go showing that no theorem using only the underlying orthogonality and low-rank graph geometry can finish the problem.

2. PARITY BALANCE AND THE SIGN ROW

For $\mu = (1^{m_1} 2^{m_2} \dots)$ put

$$z_\mu = \prod_{i \geq 1} i^{m_i} m_i!, \quad \varepsilon(\mu) = (-1)^{n - \ell(\mu)}.$$

Let $E_n = \{\mu : \varepsilon(\mu) = 1\}$ and $O_n = \{\mu : \varepsilon(\mu) = -1\}$. The sign twist is

$$(1) \quad \chi^{\lambda'}(\mu) = \varepsilon(\mu) \chi^\lambda(\mu).$$

Let $p(n)$ denote the partition number and $q(n)$ the number of self-conjugate partitions of n , equivalently the number of partitions into distinct odd parts.

Theorem 2.1 (Exact sign-row count). *For every $n \geq 0$,*

$$\sum_{\mu \vdash n} \varepsilon(\mu) = q(n), \quad N_n^-((1^n)) = \frac{p(n) - q(n)}{2}.$$

Proof. The signed generating function is

$$\sum_{n \geq 0} \left(\sum_{\mu \vdash n} \varepsilon(\mu) \right) x^n = \prod_{j \text{ odd}} \frac{1}{1 - x^j} \prod_{j \text{ even}} \frac{1}{1 + x^j}.$$

For $j = 2r$, use $(1 + x^{2r})^{-1} = (1 - x^{2r})/(1 - x^{4r})$. Cancelling the even Euler factors leaves $\prod_{r \geq 1} (1 + x^{2r-1})$, the generating function for partitions into distinct odd parts. If the sign row has P_n positive and M_n negative entries, then $P_n + M_n = p(n)$ and $P_n - M_n = q(n)$. $\square$

For a fixed row λ , let $E^-(\lambda)$ denote the number of negative values on even classes, and let $O^-(\lambda), O^0(\lambda), O^+(\lambda)$ denote the negative, zero and positive counts on odd classes.

Theorem 2.2 (Parity balance). *For every $\lambda \vdash n$,*

$$N_n^-(\lambda') - N_n^-((1^n)) = E^-(\lambda) - O^-(\lambda) - O^0(\lambda).$$

Consequently sign-row maximality at degree n is equivalent to

$$(2) \quad E^-(\lambda) \leq O^-(\lambda) + O^0(\lambda) \quad (\lambda \vdash n).$$

Proof. By (1), $N_n^-(\lambda') = E^-(\lambda) + O^+(\lambda)$. The sign row is negative precisely on O_n , so

$$N_n^-((1^n)) = |O_n| = O^-(\lambda) + O^0(\lambda) + O^+(\lambda).$$

Subtract. Conjugation permutes the irreducible rows. $\square$

The term O^0 is essential; the stronger inequality $E^- \leq O^-$ is false.

2.1. A magnitude-free representation-ring formulation. Let κ be the character of the conjugation action of S_n on itself. On a class μ , $\kappa(\mu) = z_\mu$.

Proposition 2.3 (Virtual deficit certificate). *Let $f = \chi^\lambda(1)$ and $D = (2f)!$. There are integer polynomials $P_-, P_0 \in \mathbb{Z}[x]$ satisfying*

$$P_-(t) = D\mathbf{1}_{t < 0}, \quad P_0(t) = D\mathbf{1}_{t=0} \quad (-f \leq t \leq f).$$

The virtual character

$$\Psi_\lambda = P_0(\chi^\lambda) - \varepsilon P_0(\chi^\lambda) - 2\varepsilon P_-(\chi^\lambda)$$

satisfies

$$(3) \quad \langle \Psi_\lambda, \kappa \rangle = 2D(O^-(\lambda) + O^0(\lambda) - E^-(\lambda)).$$

Hence parity balance is equivalent to $\langle \Psi_\lambda, \kappa \rangle \geq 0$.

Proof. Use Lagrange interpolation on $-f, -f+1, \dots, f$. The denominator at an interpolation node t is $(t+f)!(f-t)!$ up to sign and therefore divides $(2f)!$. Scaling by D gives integer coefficients. Polynomial expressions in a genuine character with integer coefficients are virtual characters. The values of Ψ_λ are $-2D$ on even negatives, $+2D$ on odd negatives and odd zeros, and 0 elsewhere. Since the ordinary inner product is $\langle \Phi, \Theta \rangle = \sum_{\mu \vdash n} \Phi(\mu)\Theta(\mu)/z_\mu$, substitution of $\kappa(\mu) = z_\mu$ removes the centralizer weights and gives (3). $\square$

The proposition removes all magnitude information, but Ψ_λ is virtual rather than automatically genuine. Thus the remaining issue is a subtraction/positivity problem in the representation ring.

3. SELF-CONJUGATE ROWS

We use the Murnaghan–Nakayama rule in its standard form [6, 8].

Lemma 3.1 (Hook support). *If $\chi^\lambda(\mu) \neq 0$, then every part of μ is a hook length of λ .*

Proof. Choose a part r of μ and apply Murnaghan–Nakayama with r first. A nonzero term requires a removable rim hook of length r . In beta-set language this is a bead move by r , equivalently the presence of a hook of length r . $\square$

Proposition 3.2 (Kawsathon–Rodtes). *If $\lambda = \lambda' \vdash n$ and a cycle type μ has an even part $h > n/2$, then $\chi^\lambda(\mu) = 0$.*

This is [7, Proposition 5.1].

Let

$$h_n = 2 \left(\left\lfloor \frac{n}{4} \right\rfloor + 1 \right), \quad r_n = n - h_n.$$

Then h_n is the least even integer strictly greater than $n/2$. For each $\nu \vdash r_n$, the class $(h_n) \cup \nu$ is forced zero for every self-conjugate row, and its parity is opposite to that of ν .

Proposition 3.3 (Forced even zeros). *If $\lambda = \lambda' \vdash n$ and $Z_E(\lambda)$ is the number of even classes on which χ^λ vanishes, then*

$$Z_E(\lambda) \geq \frac{p(r_n) - q(r_n)}{2}.$$

Theorem 3.4 (Self-conjugate rows satisfy OPAC). *For every $n \geq 1$ and every self-conjugate $\lambda = \lambda' \vdash n$,*

$$N_n^-(\lambda) \leq \frac{p(n) - q(n)}{2}.$$

Thus every counterexample to Conjecture 1.1 must be non-self-conjugate.

Proof. If $\lambda = \lambda'$, (1) forces every odd-class value to vanish, hence $N_n^-(\lambda) = E^-(\lambda)$. Appendix A proves

$$\frac{p(r_n) - q(r_n)}{2} \geq q(n) \quad (n \geq 13).$$

Therefore $Z_E(\lambda) \geq q(n)$ for $n \geq 13$. Since $|E_n| = (p(n) + q(n))/2$,

$$E^-(\lambda) \leq |E_n| - Z_E(\lambda) \leq \frac{p(n) - q(n)}{2}.$$

The remaining degrees $n \leq 12$ are contained in the exact verification of Section 9. $\square$

4. TWO-ROW CHARACTERS

Let $1 \leq k \leq n/2$ and $\lambda = (n - k, k)$. Write $\mu = (a_1, \dots, a_\ell)$ with indexed parts and define

$$b_j(\mu) = \# \left\{ S \subseteq \{1, \dots, \ell\} : \sum_{i \in S} a_i = j \right\}.$$

Proposition 4.1 (Subset-sum formula). *For every $\mu \vdash n$,*

$$\chi^{(n-k, k)}(\mu) = b_k(\mu) - b_{k-1}(\mu).$$

Proof. The permutation character π_j for the action of S_n on j -subsets has value $b_j(\mu)$. Young's rule gives $\pi_j = \chi^{(n)} + \chi^{(n-1, 1)} + \dots + \chi^{(n-j, j)}$, so $\pi_k - \pi_{k-1} = \chi^{(n-k, k)}$. $\square$

Write $\mu = (1^{m_1} 2^{m_2} \dots)$ and define

$$r_j(\mu) = \min_{1 \leq i \leq j} m_i, \quad D(\mu) = \sum_{j \geq 1} j r_j(\mu).$$

Theorem 4.2 (Staircase-core positivity). *If $\chi^{(n-k, k)}(\mu) < 0$, then $D(\mu) < 2k$.*

Proof. The subset-sum polynomial is $B_\mu(x) = \prod_{j \geq 1} (1 + x^j)^{m_j}$. For each multiplicity level $a \geq 1$, put $M_a = \max\{M : m_1, \dots, m_M \geq a\}$. Then

$$K_\mu(x) = \prod_{a \geq 1} \prod_{j=1}^{M_a} (1 + x^j)$$

divides $B_\mu(x)$ and has degree $D(\mu)$. Each factor $\prod_{j=1}^M (1 + x^j)$ has symmetric unimodal coefficients, and products preserve symmetric unimodality; see [12]. Write

$$K_\mu(x) = \sum_t u_t x^t, \quad B_\mu(x) = K_\mu(x) A_\mu(x), \quad A_\mu(x) = \sum_s a_s x^s,$$

with $a_s \geq 0$. By Proposition 4.1,

$$\chi^{(n-k, k)}(\mu) = [x^k](1 - x)B_\mu(x) = \sum_s a_s (u_{k-s} - u_{k-s-1}).$$

If $2k \leq D(\mu)$, every relevant index lies at or before the midpoint of the symmetric unimodal sequence (u_t) , so every summand is nonnegative. $\square$

Theorem 4.3 (Uniform two-row strip). *For every $\varepsilon > 0$, for all sufficiently large n and every*

$$1 \leq k \leq \left(\frac{\sqrt{6}}{4\pi} - \varepsilon \right) \sqrt{n} \log n,$$

one has $N_n^-((n - k, k)) < (p(n) - q(n))/2$. In fact $N_n^-((n - k, k)) = o(p(n))$ uniformly in this range.

Proof. By Theorem 4.2, negativity implies $D(\mu) < 2k$. Proposition B.1 in Appendix B gives, for a uniformly random partition $\mu \vdash n$,

$$D(\mu) \geq \left(\frac{\sqrt{6}}{2\pi} - o_{\mathbb{P}}(1) \right) \sqrt{n} \log n.$$

Hence the proportion of partitions with $D(\mu) < 2k$ tends to zero uniformly in the stated range. Since $q(n) = o(p(n))$, the sign-row target is $(1/2 + o(1))p(n)$. $\square$

5. TWO EXACT BOUNDARY PAIRS

Theorem 5.1 (Standard and near-sign hooks). *For every $n \geq 2$, both $(n-1, 1)$ and $(2, 1^{n-2})$ satisfy OPAC.*

Proof. For the standard character, $\chi^{(n-1,1)}(\mu) = m_1(\mu) - 1$, so the negative classes are exactly the partitions with no part 1. Inject these into O_n as follows. If such a partition is already odd, leave it fixed. If it is even, let $r \geq 2$ be its smallest part and replace r by $(r-1, 1)$. The length changes by one and the map is injective, giving $N_n^-((n-1, 1)) \leq |O_n|$.

For the conjugate near-sign hook, the value is $\varepsilon(\mu)(m_1(\mu) - 1)$. A direct parity count gives

$$N_n^-((2, 1^{n-2})) - |O_n| = \frac{2q(n) - q(n-1) - q(n-2) - p(n-1) + p(n-2)}{2}.$$

There are elementary injections

$$q(n) \leq q(n-1) + q(n-2), \quad q(n) \leq p(n-1) - p(n-2).$$

For the first, split a distinct-odd partition according to whether it contains 1: delete 1 in the first case, and otherwise subtract 2 from its smallest part. For the second, delete 1 when present; otherwise subtract 1 from the smallest odd part. These inequalities make the displayed difference nonpositive. $\square$

Theorem 5.2 (Second two-row conjugate pair). *For every $n \geq 4$, both $(n-2, 2)$ and $(2, 2, 1^{n-4})$ satisfy OPAC.*

Proof. By Proposition 4.1,

$$\chi^{(n-2,2)}(\mu) = m_2 + \frac{m_1(m_1 - 3)}{2},$$

so negativity occurs exactly when $m_2 = 0$ and $m_1 \in \{1, 2\}$. For negative even sources with $m_1 = 1$, splitting the smallest part $a \geq 3$ into $(a-1, 1)$ maps injectively into $O^- \cup O^0$, while merging the sole 1 into the largest part maps injectively into O^0 . For sources with $m_1 = 2$, merging one 1 into the largest part maps injectively into O^- , while splitting the smallest part maps injectively into $O^+ \cup O^0$. The image families are separated by their resulting values of m_1 , and each operation has a canonical inverse. These are exactly the two parity-balance inequalities for the conjugate pair. $\square$

6. A HALF-LOGARITHMIC BOUNDARY EXCLUSION

For a cycle type μ , let $X_j(\mu)$ be the multiplicity of the part j . For $\alpha \vdash m$, a stable character polynomial may be written [10, Proposition 4.5]

$$(4) \quad P_\alpha(X) = \sum_{|\rho|+|\sigma|=m} (-1)^{\ell(\sigma)} \frac{\chi^\alpha(\rho \cup \sigma)}{z_\sigma} \prod_{j \geq 1} \binom{X_j}{m_j(\rho)},$$

with $\chi^{(n-m, \alpha)}(\mu) = P_\alpha(X_1(\mu), \dots, X_m(\mu))$ whenever $(n-m, \alpha)$ is a partition.

For $m, N \geq 0$, let $T_m(N)$ be the number of partitions of N with all parts greater than m , let

$$D_m(N) = \sum_{\substack{\nu \vdash N \\ \text{all parts} > m}} \varepsilon(\nu), \quad Q(N) = \max_{0 \leq r \leq N} q(r).$$

Lemma 6.1 (Tail abundance and parity cancellation). *For $m, N \geq 0$,*

$$|D_m(N)| \leq 2^m Q(N),$$

and, for $N > m$,

$$T_m(N) \geq \frac{p(N)}{(N+1)^m}.$$

Proof. Theorem 2.1 gives

$$\sum_{N \geq 0} D_m(N) x^N = \left(\sum_{N \geq 0} q(N) x^N \right) \prod_{j=1}^m (1 - (-1)^{j-1} x^j).$$

The last polynomial has coefficient ℓ_1 -norm at most 2^m . For the second inequality, classify a partition by the multiplicities of $1, \dots, m$. There are at most $(N+1)^m$ profiles. Deleting the small parts leaves a tail partition, and $T_m(r)$ is nondecreasing for $r > m$ by increasing the largest part by one. $\square$

Theorem 6.2 (Half-logarithmic boundary exclusion). *Fix $c < 1/2$. Uniformly over $m = m(n)$ and $\alpha \vdash m$ satisfying*

$$m \leq c \frac{\log n}{\log \log n},$$

both $(n-m, \alpha)$ and its conjugate satisfy OPAC for all sufficiently large n . Consequently every infinite sequence of counterexamples satisfies

$$(5) \quad \liminf \frac{n - \lambda_1}{\log n / \log \log n} \geq \frac{1}{2}, \quad \liminf \frac{n - \lambda'_1}{\log n / \log \log n} \geq \frac{1}{2}.$$

Proof. Choose $\eta > 0$ with $c + 3\eta < 1/2$, and put $M = n^{1/2-\eta}$ and $L = n^{1/2+\eta}$. Call μ good if $X_1(\mu) \geq M$ and $X_j(\mu) \leq L$ for $2 \leq j \leq m$. The number with $X_1 < M$ is $p(n) - p(n - \lfloor M \rfloor) = o(p(n))$. For fixed $j \geq 2$, deleting $\lfloor L \rfloor$ copies of j bounds the number with $X_j > L$ by $p(n - 2\lfloor L \rfloor)$, whose ratio to $p(n)$ is $\exp(-\Theta(n^\eta))$ by Hardy–Ramanujan [4]. Since $m = O(\log n / \log \log n)$, the bad set is $o(p(n))$.

In (4), the term $(\rho, \sigma) = ((1^m), \emptyset)$ is $f^\alpha \binom{X_1}{m} > 0$. For a non-main summand write $k = m_1(\rho)$, $s = \sum_{j \geq 2} m_j(\rho)$, and $h = |\sigma|$. Then $m - k \geq h + 2s$. On a good class,

$$\prod_j \binom{X_j}{m_j(\rho)} \leq X_1^k L^s.$$

Uniformly for $X_1 \geq M$, $\binom{X_1}{m} \geq X_1^m / (2m!)$ for large n . Using $|\chi^\alpha(\tau)| \leq f^\alpha$, every non-main term divided by the positive main term has absolute value at most

$$2m! M^{-h} \left(\frac{L}{M^2} \right)^s \leq 2m! n^{-1/2+3\eta}.$$

There are at most $(m+1)2^m$ pairs (ρ, σ) with $|\rho| + |\sigma| = m$. Thus the total relative error is at most

$$2(m+1)2^m m! n^{-1/2+3\eta}.$$

Its logarithm is $(c - 1/2 + 3\eta + o(1)) \log n \rightarrow -\infty$ by Stirling. Hence $\chi^{(n-m, \alpha)}(\mu) > 0$ on every good class for large n , uniformly in α , and therefore $N_n^-(n-m, \alpha) = o(p(n))$.

For the conjugate row, the sign of P_α depends only on the small-cycle profile. There are at most $(n+1)^m$ profiles. Lemma 6.1 gives

$$|E^- - O^-| \leq 2^m (n+1)^m Q(n).$$

We now construct an odd-zero reservoir. If $\alpha \neq (1^m)$, setting $X_1 = \dots = X_m = 0$ in (4) yields

$$P_\alpha(0, \dots, 0) = \sum_{\sigma \vdash m} (-1)^{\ell(\sigma)} \frac{\chi^\alpha(\sigma)}{z_\sigma} = (-1)^m \langle \chi^\alpha, \text{sgn} \rangle = 0.$$

Thus every partition with all parts greater than m is a zero. If $\alpha = (1^m)$, use the profile $X_1 = 1, X_2 = \dots = X_m = 0$; the exterior-power generating function shows that the coefficient of degree m vanishes. In either case the number of odd zeros is at least $(T_m(N) - |D_m(N)|)/2$ for $N \in \{n, n-1\}$. By Lemma 6.1 and Hardy–Ramanujan,

$$\log T_m(N) = 2\pi\sqrt{n/6} + o(\sqrt{n}),$$

whereas the signed imbalance is on the distinct-odd scale $\pi\sqrt{n/6} + o(\sqrt{n})$. Hence $O^0 > |E^- - O^-|$ for large n , so parity balance holds for the conjugate row as well. Applying the result to conjugate partitions gives (5). $\square$

7. THE SQUARE-CORE REDUCTION

Write

$$\mathcal{D}_n = \{\delta \vdash n : \delta \text{ has distinct odd parts}\}, \quad \mathcal{N}_n = E_n \setminus \mathcal{D}_n,$$

so $|\mathcal{D}_n| = q(n)$ and

$$|\mathcal{N}_n| = |O_n| =: s = \frac{p(n) - q(n)}{2}.$$

Let $X = (\chi^\lambda(\mu))$ be the ordinary character table and

$$Q_{\lambda, \mu} = \frac{\chi^\lambda(\mu)}{\sqrt{z_\mu}}$$

its orthogonal normalization.

7.1. The principal-hook split block. For a self-conjugate partition σ , let $\delta(\sigma) = (h_{11}(\sigma), h_{22}(\sigma), \dots)$ be its principal diagonal hook lengths. These are distinct odd integers summing to n and give the classical bijection from self-conjugate partitions to $\mathcal{D}_n$.

Lemma 7.1 (Unimodular split block). *Order self-conjugate partitions by lexicographic order on their principal-hook sequences, and order $\mathcal{D}_n$ through the same bijection. Then*

$$X_{\text{sc}, D} = (\chi^\sigma(\delta))_{\sigma=\sigma', \delta \in \mathcal{D}_n}$$

is triangular with diagonal entries ± 1 . In particular it is unimodular.

Proof. Apply Murnaghan–Nakayama successively in decreasing principal-hook length. If two principal-hook sequences first differ at position j and the column sequence has a larger j th term, then after the common larger principal hooks are removed, the remaining diagram has no hook of that length, so the value is zero. On the diagonal, each principal hook is forced; the product of rim-hook signs is ± 1 . $\square$

7.2. The square nonsplit-even core. Choose one representative from each non-self-conjugate pair $\{\rho, \rho'\}$; call the set $\mathcal{R}$. Replace each pair of normalized rows by its symmetric and antisymmetric combinations. Then the orthogonal matrix has block form

$$(6) \quad \tilde{Q} = \begin{pmatrix} V_N & V_D & 0 \\ W_N & W_D & 0 \\ 0 & 0 & U \end{pmatrix},$$

where columns are $\mathcal{N}_n, \mathcal{D}_n, O_n$, the first block row is indexed by $\mathcal{R}$, the second by self-conjugate irreducibles, and the third by antisymmetric pair rows. Thus $U \in \mathbb{R}^{s \times s}$ is orthogonal and the full even block is orthogonal.

Theorem 7.2 (Square-core theorem). *The matrix $V_N \in \mathbb{R}^{s \times s}$ is invertible. Moreover*

$$(7) \quad |\det V_N| = |\det W_D| = \frac{1}{\sqrt{\prod_{\delta \in \mathcal{D}_n} z_\delta}}.$$

Equivalently, for the ordinary character table,

$$|\det X_{\mathcal{R}, \mathcal{N}_n}| = 2^{-s/2} \sqrt{\prod_{\nu \in \mathcal{N}_n} z_\nu \prod_{\delta \in \mathcal{D}_n} z_\delta}.$$

Proof. By Lemma 7.1,

$$W_D = X_{\text{sc}, D} \text{diag}(z_\delta^{-1/2} : \delta \in \mathcal{D}_n)$$

is invertible with the displayed determinant. Complementary minors of an orthogonal matrix have equal absolute determinant, so the complementary block V_N in (6) is invertible and has the same absolute determinant. Finally

$$V_N = \sqrt{2} X_{\mathcal{R}, \mathcal{N}_n} \text{diag}(z_\nu^{-1/2} : \nu \in \mathcal{N}_n),$$

which gives the ordinary determinant. $\square$

The ordinary block is the nonsymmetric/non-split alternating-group block studied by Bessenrodt–Olsson [2, Theorem 2.3]; the normalized proof above exposes the graph geometry needed here.

Since V_N is invertible there is a unique matrix R with $V_D = V_N R$. Orthogonality gives

$$(8) \quad R = -W_N^T (W_D^T)^{-1},$$

and

$$(9) \quad V_N (I + R R^T) V_N^T = I_s.$$

Corollary 7.3 (Square-core form of parity balance). *Fix a non-self-conjugate pair $\{\lambda, \lambda'\}$ and let x_λ be the corresponding row of V_N , while y_λ is the corresponding row of U . Then*

$$E^-(\lambda) = n_-(x_\lambda) + n_-(x_\lambda R),$$

and the two members of the pair both satisfy the sign-row bound if and only if

$$(10) \quad n_-(x_\lambda) + n_-(x_\lambda R) \leq \min\{n_{\leq 0}(y_\lambda), n_{\geq 0}(y_\lambda)\}.$$

Proof. All normalization factors are positive. The even values of λ and λ' coincide, while their odd values differ by sign. Equation (8) identifies the split-even row with $x_\lambda R$, and Theorem 2.2 applied to both pair members gives the result. $\square$

7.3. An odd-central parity bridge. For the chosen orientation $\mathcal{R}$, set

$$\mathfrak{h}_{\mathcal{R}} = \sum_{\rho \in \mathcal{R}} (e_{\rho} - e_{\rho'}),$$

where e_{ρ} are primitive central idempotents. It is odd-supported and satisfies

$$\mathfrak{h}_{\mathcal{R}}^2 = 1 - \sum_{\sigma=\sigma'} e_{\sigma}, \quad \mathfrak{h}_{\mathcal{R}}^3 = \mathfrak{h}_{\mathcal{R}}.$$

Proposition 7.4 (Odd-to-nonsplit parity bridge). *In normalized class coordinates, the odd-to-even block of multiplication by $\mathfrak{h}_{\mathcal{R}}$ is $V^T U$. Its restriction*

$$\mathfrak{h}[\mathcal{N}_n, O_n] = V_N^T U$$

is invertible. Hence multiplication by $\mathfrak{h}_{\mathcal{R}}$ gives an exact linear isomorphism $\mathbb{R}^{O_n} \cong \mathbb{R}^{\mathcal{N}_n}$.

Proof. In the pair-adapted spectral basis, multiplication by $\mathfrak{h}_{\mathcal{R}}$ swaps symmetric and anti-symmetric pair coordinates and annihilates every self-conjugate coordinate. Transforming back gives $V^T U$. The restriction is invertible by Theorem 7.2 and orthogonality of U . $\square$

8. ARITHMETIC STRUCTURE OF THE SQUARE CORE

Write

$$X_N = X_{\mathcal{R}, \mathcal{N}_n}, \quad X_D = X_{\mathcal{R}, \mathcal{D}_n}, \quad X_O = X_{\mathcal{R}, O_n},$$

and let Z_N, Z_D, Z_O be the corresponding centralizer diagonal matrices.

8.1. Cross-characteristic invertibility.

Proposition 8.1. *For every prime $\ell > n$, both X_N and X_O are invertible over $\mathbb{F}_{\ell}$.*

Proof. The determinant formula in Theorem 7.2, equivalently the alternating-group determinant of Bessenrodt–Olsson, has prime support at most n , so X_N stays invertible modulo $\ell > n$. For the odd core,

$$U = \sqrt{2} X_O Z_O^{-1/2}, \quad U U^T = I,$$

whence

$$(11) \quad 2X_O Z_O^{-1} X_O^T = I, \quad X_O^{-1} = 2Z_O^{-1} X_O^T.$$

Thus no prime exceeding n divides $\det X_O$. $\square$

8.2. Integral graph lattice and determinant one.

Proposition 8.2 (Integral graph lattice). *Define*

$$C = -X_{\text{sc}, N}^T (X_{\text{sc}, D}^T)^{-1}.$$

Then $C \in M_{s \times q(n)}(\mathbb{Z})$ and

$$(12) \quad R = Z_N^{-1/2} C Z_D^{1/2}, \quad X_D = X_N Z_N^{-1} C Z_D.$$

Moreover

$$(13) \quad \det(Z_D^{-1} + C^T Z_N^{-1} C) = 1.$$

Proof. The split block $X_{\text{sc}, D}$ is unimodular by Lemma 7.1, hence C is integral. Substitution into (8) gives (12). Taking determinants in (9) and using (7) gives

$$\det(I + R R^T) = \det(I + R^T R) = \prod_{\delta \in \mathcal{D}_n} z_{\delta} = \det Z_D.$$

But (12) gives

$$I + R^T R = Z_D^{1/2} (Z_D^{-1} + C^T Z_N^{-1} C) Z_D^{1/2},$$

so cancellation yields (13). $\square$

8.3. Low-rank distortion and the phase normal form. Put

$$M = I + RR^T, \quad H = V_N^T U.$$

Proposition 8.3 (Arithmetic phase normal form). *One has*

$$\begin{aligned} V_N^T V_N &= M^{-1}, & V_N^{-1} &= M V_N^T, \\ HH^T &= M^{-1}, & I - HH^T &= R(I + R^T R)^{-1} R^T \succeq 0. \end{aligned}$$

Hence at least $s - q(n)$ singular values of the canonical odd/nonsplit-even bridge are exactly 1. If $P = V_N^{-1} U$, then $PP^T = M$, so there are orthogonal matrices $O, S \in O(s)$ with

$$V_N = OM^{-1/2}, \quad P = M^{1/2}S.$$

For corresponding rows, writing z for the appropriate row of O ,

$$(14) \quad x = zM^{-1/2}, \quad xR = zM^{-1/2}R, \quad y = zS.$$

Proof. Invert (9). Since $UU^T = I$, $HH^T = V_N^T V_N = M^{-1}$, and the Woodbury identity gives

$$I - M^{-1} = R(I + R^T R)^{-1} R^T.$$

Finally $PP^T = V_N^{-1} V_N^{-T} = M$, so polar decomposition gives the stated orthogonal factors. $\square$

Thus all real metric distortion is confined to at most $q(n)$ directions; the unresolved sign problem is to control the phase S by arithmetic or genuine character-theoretic information.

8.4. A canonical mod-2 relation. Let Π be the permutation matrix of the standard binary split/merge pairing between $\mathcal{N}_n$ and O_n : along the least nonterminal 2-adic chain of parts, merge two equal parts $2^r d$ into one part $2^{r+1} d$, or perform the inverse split. The fixed points on all partitions are precisely the distinct-odd partitions, so the restriction pairs $\mathcal{N}_n$ bijectively with O_n .

Proposition 8.4 (Binary pairing congruence).

$$(15) \quad X_N \equiv X_O \Pi \pmod{2}.$$

Proof. Miller [9, Proposition 1] proves that replacing p equal k -cycles by one pk -cycle preserves every S_n character value modulo p . Every edge of the binary involution is such a merge/split with $p = 2$. $\square$

8.5. Generic real square-core geometry is insufficient.

Proposition 8.5 (Real-metric no-go). *The real identities consisting of orthogonality, $V_D = V_N R$, $V_N(I + RR^T)V_N^T = I$, the determinant constraint and an arbitrary orthogonal odd core do not imply the square-core sign inequality (10).*

Proof. Take

$$\begin{aligned} R &= \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}, & M &= \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, & O &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \\ V_N &= OM^{-1/2}, & U &= O \operatorname{diag}(1, -1). \end{aligned}$$

The first corresponding row has

$$x = (1/\sqrt{6}, -1/\sqrt{2}), \quad xR = 1/\sqrt{3} > 0, \quad y = (1/\sqrt{2}, 1/\sqrt{2}).$$

Thus the left side of (10) is 1 while the right side is 0. The even rows can be completed to a full orthogonal block by $(-\sqrt{2/3}, 0, 1/\sqrt{3})$. Hence the real square-core package alone is insufficient; arithmetic constraints such as (12), (13) and (15), or other actual S_n structure, are essential. $\square$

Remark 8.6 (Direct remaining target). Equations (14) and (10) isolate the unresolved non-self-conjugate problem in the form

$$n_{-}(zM^{-1/2}) + n_{-}(zM^{-1/2}R) \leq \min\{n_{\leq 0}(zS), n_{\geq 0}(zS)\}.$$

The direct target is therefore an *arithmetic phase theorem*: prove this inequality for the actual symmetric-group phase S using the simultaneous integral graph lattice, determinant-one Frobenius form, cross-characteristic invertibility, mod-2 pairing, or further genuine character-table structure. Proposition 8.5 shows that generic real geometry cannot supply the missing implication.

9. EXACT COMPUTATION

The OPAC inequality has been verified in exact arithmetic for every $1 \leq n \leq 29$. The sign row is the unique maximizer at $n = 2, 6$ and every $8 \leq n \leq 29$; non-sign ties occur only at $n = 3, 4, 5, 7$. Character tables were computed by integer Murnaghan–Nakayama recursion and each class column was independently checked by

$$\sum_{\lambda \vdash n} \chi^{\lambda}(\mu)^2 = z_{\mu}.$$

The same exact arithmetic supplies the finite cases used in Theorem 3.4. No computational observation is used in the asymptotic arguments.

Remark 9.1 (Exploratory full-rank phenomenon). An unarchived exact/modular pass reports

$$\text{rank } X_{\text{sc}, \mathcal{N}_n} = q(n) \quad (3 \leq n \leq 35).$$

This is not a theorem and is not used above. If true in all degrees, then C and R have full column rank $q(n)$ and the bridge in Proposition 8.3 has exactly $q(n)$ distorted singular directions. Archival certificates are required before this observation should be promoted beyond CHECK status.

10. CONCLUSION

The sign-row problem is reduced exactly to parity balance. Self-conjugate rows are settled in every degree; a uniform $\sqrt{n} \log n$ strip of two-row characters is settled; the first two nontrivial conjugate boundary pairs are settled in every degree; and Theorem 6.2 excludes every infinite counterexample sequence from a boundary layer of asymptotic width $(1/2 - o(1)) \log n / \log \log n$.

For the remaining non-self-conjugate rows, Sections 7 and 8 isolate a square nonsplit-even core and an odd core of the same dimension. The split-even sector is encoded by the integral Frobenius-scaled graph matrix C , whose rational Frobenius Gram form has determinant one. Both square cores survive modulo every prime $\ell > n$, the parity bridge is an isometry off at most $q(n)$ directions, and Miller’s congruence ties the two integral bases together modulo 2. Proposition 8.5 proves that the associated real metric geometry is not enough by itself. The direct unresolved step is therefore an arithmetic sign theorem for the actual symmetric-group phase in (14). Proposition 2.3 gives a complementary representation-ring formulation in which the OPAC deficit is an exact integral pairing with the conjugation character. The global conjecture remains open.

APPENDIX A. EFFECTIVE COMPLETION OF THE SELF-CONJUGATE SECTOR

Lemma A.1 (Distinct-odd bound). *For every $n \geq 1$,*

$$q(n) \leq e^{\pi \sqrt{n/6}}.$$

Proof. Write $Q(x) = \prod_{j \geq 1} (1 + x^{2j-1})$ and $f(x) = \log(1 + e^{-x})$. The function f is convex. Midpoint convexity on $[(2j-2)t, 2jt]$ gives

$$2tf((2j-1)t) \leq \int_{(2j-2)t}^{2jt} f(x) dx.$$

Summing and using $\int_0^\infty \log(1 + e^{-x}) dx = \pi^2/12$ yields $\log Q(e^{-t}) \leq \pi^2/(24t)$. Since $q(n)e^{-tn} \leq Q(e^{-t})$,

$$\log q(n) \leq nt + \frac{\pi^2}{24t}.$$

Minimize at $t = \pi/\sqrt{24n}$. □

Proposition A.2 (Effective partition comparison). *For every $n \geq 13$,*

$$\frac{p(r_n) - q(r_n)}{2} \geq q(n), \quad r_n = n - 2 \left(\left\lfloor \frac{n}{4} \right\rfloor + 1 \right).$$

Proof. For $13 \leq n \leq 249$ the inequality was checked in exact integer arithmetic using independent recurrences for p and q . For $n \geq 250$, the explicit lower bound of Banerjee–Paule–Radu–Zeng [1, Theorem 1.3] gives

$$p(m) > \frac{e^{\pi\sqrt{2m/3}}}{4\sqrt{3}m} \left(1 - \frac{1}{2\sqrt{m}} \right).$$

Since $n/2 - 2 \leq r_n < n/2$,

$$p(r_n) > \frac{3}{8\sqrt{3n}} e^{\pi\sqrt{(n-4)/3}}.$$

By Lemma A.1, $q(r_n) + 2q(n) \leq 3e^{\pi\sqrt{n/6}}$. It is enough to verify

$$\pi \left(\sqrt{\frac{n-4}{3}} - \sqrt{\frac{n}{6}} \right) > \log(8\sqrt{3n}).$$

The difference is positive at $n = 250$ and has positive derivative thereafter. □

APPENDIX B. RANDOM-PARTITION SIZE OF THE STAIRCASE CORE

Under Fristedt’s Boltzmann measure [3] with parameter $q = e^{-t}$, the multiplicities Z_i are independent geometric random variables with

$$\mathbb{P}(Z_i \geq \ell) = e^{-t\ell}.$$

Let

$$r_j = \min_{1 \leq i \leq j} Z_i, \quad S_j = \frac{j(j+1)}{2}.$$

Then $\mathbb{P}(r_j \geq \ell) = e^{-tS_j\ell}$ and $\mathbb{E}r_j = (e^{tS_j} - 1)^{-1}$. Consequently

$$(16) \quad \mathbb{E}D = \sum_{j \geq 1} \frac{j}{e^{tS_j} - 1} = \frac{1}{t} \log \frac{1}{t} + O(t^{-1}).$$

For $j \leq k$, the nested-minimum memoryless structure gives

$$\text{Cov}(r_j, r_k) = \text{Var}(r_k) = \frac{e^{-tS_k}}{(1 - e^{-tS_k})^2},$$

and summing covariances yields

$$(17) \quad \text{Var } D = \sum_{k \geq 1} k^3 \frac{e^{-tS_k}}{(1 - e^{-tS_k})^2} = \frac{2}{t^2} \log \frac{1}{t} + O(t^{-2}).$$

Thus $\sqrt{\text{Var } D}/\mathbb{E}D = O((\log(1/t))^{-1/2})$.

For a uniform random partition of n , take $t_n = \pi/\sqrt{6n}$ and $J_n = t_n^{-1/2}/\log(1/t_n)$. The truncated statistic $D_{J_n} = \sum_{j \leq J_n} jr_j$ depends only on multiplicities of part sizes $j \leq J_n = o(\sqrt{n})$. Fristedt's conditioning device, in the quantitative form developed by Pittel [11], transfers these coordinates from the Boltzmann model to the uniform partition model. The truncation retains the leading terms of (16) and (17). Chebyshev's inequality gives:

Proposition B.1. *For a uniformly random partition $\mu \vdash n$,*

$$D(\mu) \geq \left(\frac{\sqrt{6}}{2\pi} - o_{\mathbb{P}}(1) \right) \sqrt{n} \log n.$$

COMPUTATIONAL REPRODUCIBILITY

The exact finite verification and the recurrence checks used in Proposition A.2 were performed in integer arithmetic. Supplementary code and certificates are archived at doi:10.5281/zenodo.22076808. The exploratory rank statement in Remark 9.1 is explicitly not part of the archived theorem certificate.

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