Direct Progress on OPAC-038: Negative Character Values in Symmetric Groups

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Abstract

For an irreducible character χ^λ of the symmetric group S_n, let

$$N_n^-(\lambda) = \#\{\mu \vdash n : \chi^\lambda(\mu) < 0\},$$

where conjugacy classes are counted uniformly. OPAC-038 asks whether, for all sufficiently large n, this quantity is maximized by the sign representation. If $p(n)$ is the partition number and $q(n)$ is the number of self-conjugate partitions, then the sign row has exactly $(p(n) - q(n))/2$ negative entries.

We prove several direct results toward this extremal problem. First, we give exact parity and sign-twist formulations and an exact positive-semidefinite fusion filter whose nullity is the number of negative entries. Second, we prove the desired inequality in every degree for the standard representation $(n-1, 1)$ and for the two-row family $(n-2, 2)$ by explicit reversible injections from negative conjugacy classes into odd conjugacy classes. Third, for every fixed partition ν, we prove that the stable family $(n - |\nu|, \nu)$ satisfies the much stronger asymptotic estimate $N_n^-/p(n) \to 0$. Thus any infinite counterexample family must leave every fixed-tail regime. The global problem remains open; the results isolate the genuinely difficult regime to partitions whose distance from the first row grows with n.

1 Introduction

Irreducible characters and conjugacy classes of S_n are both indexed by partitions of n. We write $\chi^\lambda(\mu)$ for the irreducible character indexed by $\lambda \vdash n$ evaluated on the class of cycle type $\mu \vdash n$. Character values are integers and may be computed by the Murnaghan–Nakayama rule; see Murnaghan's original treatment [6] and modern work on character-table vanishing such as [1, 3].

The statistic of interest is deliberately unweighted:

$$N_n^-(\lambda) = \#\{\mu \vdash n : \chi^\lambda(\mu) < 0\}.$$

The extremal question called OPAC-038 asks whether the sign representation (1^n) maximizes $N_n^-(\lambda)$ for all sufficiently large n. This question belongs to a broader study of parity, zeros, and large-scale behavior in symmetric-group character tables; see [4, 5, 8].

Let $p(n)$ be the number of partitions of n, and let $q(n)$ be the number of self-conjugate partitions of n, equivalently the number of partitions into distinct odd parts. Put

$$r(n) = \frac{p(n) - q(n)}{2}, \qquad h(n) = \frac{p(n) + q(n)}{2}.$$

Our main results are the following.

Theorem 1.1 (Exact direct families)**.** *For every $n \geq 2$,*

$$N_n^-(n-1,1) \leq r(n).$$

For every $n \geq 4$,

$$N_n^-(n-2,2) \leq r(n).$$

Both inequalities are proved by explicit injections from the negative classes of the indicated row into the odd conjugacy classes of S_n.

Theorem 1.2 (Stable fixed-tail families)**.** *Fix a partition $\nu \vdash k$. For $n \geq k+\nu_1$, let $\lambda[n] = (n-k, \nu)$. Then*

$$\frac{N_n^-(\lambda[n])}{p(n)} \longrightarrow 0 \qquad (n \to \infty).$$

Consequently $N_n^-(\lambda[n]) < r(n)$ for all sufficiently large n.

Theorem 1.2 shows that a hypothetical infinite counterexample sequence cannot remain a bounded distance below the first row. This complements the exact injections in Theorem 1.1, which show that direct class surgery can prove the conjectured bound uniformly in n for nontrivial infinite families.

2 The sign row and parity

For a cycle type μ, let $\ell(\mu)$ denote its number of parts and set

$$\varepsilon(\mu) = (-1)^{n-\ell(\mu)}.$$

Thus $\varepsilon(\mu)$ is the sign of a permutation of cycle type μ.

Proposition 2.1 (Sign-row count)**.** *For every n,*

$$\sum_{\mu \vdash n} \varepsilon(\mu) = q(n), \qquad N_n^-(1^n) = r(n).$$

Proof. The signed generating function is

$$\sum_{n \geq 0} \left(\sum_{\mu \vdash n} \varepsilon(\mu) \right) x^n = \prod_{j \text{ odd}} (1-x^j)^{-1} \prod_{j \text{ even}} (1+x^j)^{-1} = \prod_{a \geq 1} (1+x^{2a-1}),$$

the generating function for partitions into distinct odd parts, hence for self-conjugate partitions. If E_n and O_n are the numbers of even and odd conjugacy classes, then $E_n + O_n = p(n)$ and $E_n - O_n = q(n)$. Therefore $O_n = r(n)$, and the sign row is negative exactly on those O_n classes. $\square$

The sign twist gives the basic symmetry

$$\chi^{\lambda'}(\mu) = \varepsilon(\mu)\chi^{\lambda}(\mu). \tag{1}$$

For a fixed row, write E_- for the number of negative values on even classes and O_-, O_0, O_+ for the negative, zero, and positive values on odd classes.

Proposition 2.2 (Exact conjugate-pair form)**.** *For every $\lambda \vdash n$,*

$$N_n^-(\lambda) = E_- + O_-, \qquad N_n^-(\lambda') = E_- + O_+.$$

Hence the conjectured bound for both λ and λ' is exactly

$$E_- + \max\{O_-, O_+\} \le r(n). \tag{2}$$

If $\lambda = \lambda'$, then χ^λ vanishes on every odd class.

Proof. The first two identities are immediate from (1). If $\lambda = \lambda'$, then on an odd class (1) gives $\chi^\lambda(\mu) = -\chi^\lambda(\mu)$. □

3 Two direct all-degree injections

We now prove Theorem 1.1. The target set in both constructions is the set of odd conjugacy classes, which has cardinality $r(n)$ by Proposition 2.1.

3.1 The standard representation

The standard character satisfies

$$\chi^{(n-1,1)}(\mu) = m_1(\mu) - 1,$$

where $m_j(\mu)$ denotes the multiplicity of the part j in μ. Hence the negative classes are precisely the partitions with no part 1.

Theorem 3.1. *For every $n \ge 2$, there is an injection from*

$$\{\mu \vdash n : m_1(\mu) = 0\}$$

into the odd conjugacy classes. Consequently $N_n^-(n-1,1) \le r(n)$.

Proof. Let μ have no part 1. If μ is already odd, set $F(\mu) = \mu$. Otherwise let $s \ge 2$ be the smallest part of μ. If $s = 2$, replace one part 2 by $1 + 1$. If $s \ge 3$, replace one part s by $(s-1) + 1$. In either case the number of parts increases by one, so the permutation parity changes and $F(\mu)$ is odd.

The unchanged images contain no 1, whereas every modified image contains a 1, so these image sets are disjoint. A modified image with at least two 1's must come from the case $s = 2$ and is inverted by merging two 1's. Otherwise it has exactly one 1; its smallest non-1 part is $s - 1$, strictly smaller than every untouched source part, so merging that part with the unique 1 recovers s. Thus F is injective. □

3.2 The family $(n-2,2)$

Stable character-polynomial theory gives, and direct character calculation confirms in the full range $n \ge 4$,

$$\chi^{(n-2,2)}(\mu) = \binom{m_1(\mu)}{2} + m_2(\mu) - m_1(\mu). \tag{3}$$

Character polynomials for stable irreducible families are treated systematically in [7].

Lemma 3.2 (Negative locus)**.** *For $n \ge 4$,*

$$\chi^{(n-2,2)}(\mu) < 0 \quad \Longleftrightarrow \quad m_2(\mu) = 0 \;\; \textit{and} \;\; m_1(\mu) \in \{1, 2\}.$$

Proof. If $m_1 = 0$, (3) equals $m_2 \geq 0$. For $m_1 = 1$ or 2 it equals $m_2 - 1$. For $m_1 \geq 3$, the term $\binom{m_1}{2} - m_1 = m_1(m_1 - 3)/2$ is nonnegative. $\square$

Theorem 3.3. *For every $n \geq 4$, the negative classes of $\chi^{(n-2,2)}$ inject into the odd conjugacy classes. Hence*

$$N_n^-(n-2,2) \leq r(n).$$

Proof. Let μ satisfy Lemma 3.2. If μ is already odd, set $G(\mu) = \mu$. Suppose μ is even.

If $m_1 = 2$, replace the two 1's by a single 2. The number of parts decreases by one, so the image is odd, with $(m_1, m_2) = (0, 1)$.

If $m_1 = 1$, there is no part 2. Let $s \geq 3$ be the smallest non-1 part and replace

$$s \longmapsto 2 + (s-2).$$

The number of parts increases by one, so again the image is odd. If $s = 3$ the image has $(m_1, m_2) = (2, 1)$; if $s = 4$ it has $(1, 2)$; and if $s \geq 5$ it has $(1, 1)$. These image types are disjoint from one another and from the unchanged odd sources, which retain $m_2 = 0$.

Each case is reversible. Type $(0, 1)$ is inverted by $2 \mapsto 1 + 1$. Type $(2, 1)$ is inverted by merging the 2 with one 1 into 3. Type $(1, 2)$ is inverted by merging the two 2's into 4. Finally, in type $(1, 1)$ the newly created part $s - 2 \geq 3$ is the unique smallest part at least 3, since every untouched source part is at least s; merge it with the 2 to recover s. Thus G is injective. $\square$

4 All fixed-tail families are asymptotically safe

We next prove Theorem 1.2. For fixed $\nu \vdash k$, stable irreducible characters are represented by a character polynomial $P_\nu(X_1, \ldots, X_k)$ satisfying

$$\chi^{(n-k,\nu)}(\mu) = P_\nu(m_1(\mu), \ldots, m_k(\mu))$$

for $n \geq k + \nu_1$; see [7].

Give X_j weighted degree j. The polynomial P_ν has weighted degree k. At the identity, $X_1 = n$ and $X_j = 0$ for $j \geq 2$. The hook-length formula gives

$$\dim(n-k, \nu) = \frac{f^\nu}{k!} n^k + O(n^{k-1}),$$

so the coefficient of X_1^k is $f^\nu / k! > 0$. Every other monomial of weighted degree at most k has ordinary total degree at most $k - 1$. Therefore

$$P_\nu(X) = \frac{f^\nu}{k!} X_1^k + R_\nu(X), \qquad \deg_{\text{ordinary}} R_\nu \leq k - 1. \tag{4}$$

We use only elementary consequences of partition asymptotics. Let ${}_n$ be uniformly distributed over partitions of n, and write $M_j = m_j({}_n)$. For fixed j and integer $L \geq 0$,

$$\Pr(M_j \geq L) \leq \frac{p(n - jL)}{p(n)},$$

while for $j = 1$ equality holds. Hardy–Ramanujan asymptotics [2] imply that, for any fixed $0 < \eta < 1/2$,

$$\Pr(M_1 < n^{1/2-\eta}) \longrightarrow 0, \tag{5}$$

and, for every fixed j and every $\eta > 0$,

$$\Pr(M_j > n^{1/2+\eta}) \longrightarrow 0. \tag{6}$$

Indeed these follow by applying the asymptotic ratio $p(n-t)/p(n)$ in the regimes $t = o(n)$ relevant here.

Proof of Theorem 1.2. Choose $\eta > 0$ so small that

$$k(1/2 - \eta) > (k-1)(1/2 + \eta),$$

for example any $\eta < 1/(4(2k-1))$. With probability tending to one, (5)–(6) give

$$M_1 \geq n^{1/2-\eta}, \qquad M_j \leq n^{1/2+\eta} \quad (2 \leq j \leq k).$$

On this event, the positive leading term in (4) is at least a positive constant times $n^{k(1/2-\eta)}$, whereas the remainder is $O(n^{(k-1)(1/2+\eta)})$. The exponent inequality therefore implies $P_\nu(M_1, \dots, M_k) > 0$ for all sufficiently large n on an event whose probability tends to one. Hence

$$\frac{N_n^-(n-k, \nu)}{p(n)} = \Pr\big(P_\nu(M_1, \dots, M_k) < 0\big) \longrightarrow 0.$$

Finally, $q(n)/p(n) \to 0$ by the classical partition asymptotics, so $r(n)/p(n) \to 1/2$. The desired eventual inequality follows. □

Corollary 4.1. *If $\lambda(n) \vdash n$ violates $N_n^-(\lambda(n)) \leq r(n)$ for infinitely many n, then $n - \lambda_1(n)$ is unbounded along that subsequence.*

5 An exact positive filter

Although the preceding results are direct sign-count arguments, it is useful to retain one exact global reformulation. Let $f = \chi^\lambda(1)$ and define

$$Q_f(x) = \prod_{j=1}^{f} (x + j).$$

All irreducible character values are integers in $[-f, f]$, so on the character spectrum

$$Q_f(x) = 0 \iff x < 0, \qquad Q_f(x) > 0 \iff x \geq 0.$$

Let K_λ denote the matrix of tensor multiplication by the irreducible S_n-module V_λ in the irreducible basis; its entries are Kronecker coefficients. It is diagonalized by the character table, with eigenvalues $\chi^\lambda(\mu)$.

Proposition 5.1 (Positive fusion filter)**.** *The matrix*

$$F_\lambda = Q_f(K_\lambda)$$

is symmetric, integral, entrywise nonnegative, and positive semidefinite, and

$$\operatorname{nullity} F_\lambda = N_n^-(\lambda).$$

Equivalently, OPAC for λ is the rank inequality

$$\operatorname{rank} F_\lambda \geq h(n).$$

Proof. The polynomial Q_f is the character of the genuine representation

$$\bigotimes_{j=1}^{f} (V_\lambda \oplus \mathbf{1}^{\oplus j}),$$

so F_λ is its multiplication matrix and is entrywise nonnegative and integral. Character-table diagonalization gives eigenvalues $Q_f(\chi^\lambda(\mu))$, all nonnegative, and they vanish exactly on negative character values. Hence F_λ is positive semidefinite and has the stated nullity. $\square$

Proposition 5.1 removes character magnitudes exactly, but it does not by itself prove the required rank lower bound. The direct injections above avoid this remaining linear-algebra problem for two infinite families.

6 Discussion and remaining problem

The sign representation is not merely a convenient benchmark: Proposition 2.1 shows that it attains the proposed bound exactly in every degree. Theorems 3.1 and 3.3 show that explicit parity-changing class surgery can prove the same extremal inequality for nontrivial infinite families. Theorem 1.2 goes further asymptotically: every fixed-tail family has a vanishing proportion of negative entries, while the sign row has asymptotic negative density 1/2.

These results isolate the unresolved regime. A counterexample family, if one exists, must have a growing number of boxes below the first row. The natural next direct target is the other two-box tail,

$$\chi^{(n-2,1,1)}(\mu) = \frac{(m_1-1)(m_1-2)}{2} - m_2,$$

whose negative locus is still a two-variable integer region but is no longer bounded in m_2. More generally, one may ask whether reversible parity-changing surgeries can be constructed systematically from the negative regions of stable character polynomials.

The global OPAC-038 problem remains open. The principal challenge is to obtain a uniform argument when the tail size $n-\lambda_1$ grows with n, where fixed-degree character-polynomial dominance no longer applies directly.

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References

[1] E. Giannelli, S. Law and E. McDowell, *Degrees and prime power order zeros of characters of symmetric and alternating groups*, Bull. Lond. Math. Soc. **57** (2025), 3408–3428.

[2] G. H. Hardy and S. Ramanujan, *Asymptotic formulae in combinatory analysis*, Proc. London Math. Soc. (2) **17** (1918), 75–115.

[3] M. L. Lewis, L. Morotti, E. Pacifici, L. Sanus and H. P. Tong-Viet, *On common zeros of characters of finite groups*, Algebr. Represent. Theory **28** (2025), 395–406.

[4] A. R. Miller, *On parity and characters of symmetric groups*, J. Combin. Theory Ser. A **162** (2019), 231–240.

[5] A. R. Miller and D. Scheinerman, *Large-scale Monte Carlo simulations for zeros in character tables of symmetric groups*, Math. Comp. **94** (2025), 505–515.

[6] F. D. Murnaghan, *The characters of the symmetric group*, Amer. J. Math. **59** (1937), 739–753.

[7] R. C. Orellana and M. Zabrocki, *Symmetric group characters as symmetric functions*, Adv. Math. **390** (2021), 107943.

[8] S. Peluse and K. Soundararajan, *Divisibility of character values of the symmetric group by prime powers*, Algebra Number Theory **19** (2025), 365–382.