

SCHUR-POSITIVE CLASS SUMS, STRICT CONSTITUENT POSITIVITY, AND CHARACTER INEQUALITIES FOR SYMMETRIC GROUPS

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ABSTRACT. Let T be a set of conjugacy classes of S_n such that $P_T = \sum_{\mu \in T} p_\mu$ is Schur-positive, and let Φ_T be the corresponding genuine character. We record the exact representation-ring matrix attached to this positivity. Multiplication by $[\Phi_T]$ is symmetric, positive semidefinite, entrywise nonnegative and integral; its class eigenvalues are z_μ on T and 0 off T , so its rank is $|T|$. More generally, for every irreducible χ^λ and $j \geq 1$, entrywise nonnegativity of powers of the multiplication matrix for $\Phi_T \chi^\lambda$ gives

$$\sum_{\mu \in T} z_\mu^{j-1} \chi^\lambda(\mu)^j \chi^\alpha(\mu) \chi^\beta(\mu) \geq 0$$

for all $\alpha, \beta \vdash n$. Specializing to Sundaram's Schur-positive set of even non-split classes makes the $j = 1$ inequality completely unweighted and forces the matrices through the partition-conjugation invariant quotient. We distinguish throughout the classical character-table diagonalization and Sundaram's Schur-positivity input from the constituent-level inequalities obtained by retaining individual matrix entries. Exact rank and kernel formulas and explicit S_3 and S_4 examples are included. We further prove a strict-positivity theorem: whenever the selector character Φ_T contains every irreducible, every unweighted constituent-level triple sum is strictly positive. Applied to Sundaram's even non-split selector, this yields strict positivity for all triples of irreducible characters. We conclude with a quantitative lower-bound conjecture supported by exact computation through $n = 13$.

1. INTRODUCTION

The character table of a finite group diagonalizes multiplication in its complexified representation ring. For S_n this standard fact becomes especially concrete under the Frobenius characteristic map. If a sum of power sums

$$P_T = \sum_{\mu \in T} p_\mu$$

is Schur-positive, it is the Frobenius characteristic of a genuine character Φ_T satisfying $\Phi_T(\mu) = z_\mu$ on T and 0 elsewhere. Multiplication by Φ_T therefore has an immediately visible spectrum.

The diagonalization itself is classical, and the relevant Schur-positive class sets used below are due to Sundaram [5]. In particular, her Theorem 1.1(3) (equivalently the corresponding result developed later in that paper) includes the set of even cycle types after the distinct-odd types are removed. We do not claim either ingredient as new. The point of this note is to retain the *individual matrix entries* after combining such a class-set character with Kronecker multiplication. This produces a family of constituent-level inequalities that is invisible if one immediately takes traces or row sums.

For a class function f let $K(f)$ denote multiplication by f in the irreducible basis. If P_T is Schur-positive, then $L_T := K(\Phi_T)$ has eigenvalues z_μ for $\mu \in T$ and 0 otherwise. Hence L_T is

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positive semidefinite, has rank $|T|$, and is entrywise nonnegative because its entries are tensor-product multiplicities. This matrix statement is a direct reformulation of the Schur-positive class sum.

The additional observation is that for every $\lambda \vdash n$

$$M_T(\lambda) := L_T K(\chi^\lambda) = K(\Phi_T \chi^\lambda)$$

is again an entrywise-nonnegative integral matrix, although it need not be positive semidefinite. Since powers remain entrywise nonnegative, reconstructing their entries from the character-table eigenbasis yields, for $j \geq 1$,

$$H_{T,j}(\alpha, \beta; \lambda) := \sum_{\mu \in T} z_\mu^{j-1} \chi^\lambda(\mu)^j \chi^\alpha(\mu) \chi^\beta(\mu) = (M_T(\lambda)^j)_{\alpha\beta} \geq 0.$$

This is valid for every Schur-positive class subset T . When $T = B$ is Sundaram's even non-split set, the case $j = 1$ loses all centralizer weights:

$$\sum_{\mu \in B} \chi^\lambda(\mu) \chi^\alpha(\mu) \chi^\beta(\mu) \geq 0.$$

The labels α, β survive, so this is a matrix of inequalities rather than a scalar row-sum statement.

Because every class in B is even, sign conjugation identifies the α and α' rows and columns. Consequently the whole hierarchy annihilates the anti-invariant sector for partition conjugation and factors through the invariant quotient. We also record the exact rank of $M_T(\lambda)$, which counts selected classes on which χ^λ is nonzero.

The paper is intentionally conservative about novelty. Section 2 reviews the standard diagonalization and the immediate matrix consequence of Schur positivity. Section 3 proves the general constituent hierarchy. Section 4 specializes to Sundaram's even non-split set and develops the sign-conjugation quotient and an S_3 example. Section 5 separates formal consequences from the genuinely additional entrywise information and records limitations.

Novelty boundary. Two ingredients used below are classical. First, the irreducible character table diagonalizes multiplication in the complexified representation ring. Second, Sundaram proved Schur positivity for several power-sum sums attached to distinguished unions of conjugacy classes. We do not claim either fact as new. The purpose of the paper is to retain individual matrix coefficients after these ingredients are combined. This produces, uniformly in three irreducible labels and in every tensor power, inequalities that are lost as soon as one takes a trace, a row sum, or the trivial-isotypic multiplicity. The general statements are therefore formulated so that the exact dependence on the chosen Schur-positive class set remains visible.

2. SCHUR-POSITIVE CLASS SUMS AS MULTIPLICATION MATRICES

Let $\mathcal{P}(n)$ be the partitions of n and write

$$X = (\chi^\alpha(\mu))_{\alpha, \mu \in \mathcal{P}(n)}, \quad Z = \text{diag}(z_\mu)_{\mu \vdash n}.$$

Character orthogonality gives

$$X^\top X = Z, \quad X Z^{-1} X^\top = I.$$

For a class function f , define

$$K(f)_{\alpha\beta} = \langle f \chi^\alpha, \chi^\beta \rangle = \sum_{\mu \vdash n} \frac{f(\mu) \chi^\alpha(\mu) \chi^\beta(\mu)}{z_\mu}.$$

Thus

$$K(f) = X Z^{-1} \text{diag}(f(\mu)) X^\top,$$

and the normalized character-table columns $z_\mu^{-1/2}(\chi^\alpha(\mu))_\alpha$ are an orthonormal eigenbasis with eigenvalue $f(\mu)$.

Under Frobenius characteristic,

$$\text{ch}(\Phi) = \sum_{\mu \vdash n} \frac{\Phi(\mu)}{z_\mu} p_\mu.$$

Hence if $P_T = \sum_{\mu \in T} p_\mu$ is Schur-positive and Φ_T is the corresponding genuine character, then

$$\Phi_T(\mu) = z_\mu \mathbf{1}_T(\mu).$$

Proposition 2.1 (Matrix form of a Schur-positive class sum). *Suppose P_T is Schur-positive. Then $L_T := K(\Phi_T)$ is symmetric, positive semidefinite, entrywise nonnegative and integral, with*

$$\text{Spec}(L_T) = \{z_\mu : \mu \in T\} \cup \{0 : \mu \notin T\}.$$

In particular,

$$\text{rank } L_T = |T|,$$

and its range and kernel are respectively the spans of the normalized character-table columns indexed by T and its complement.

Proof. The spectral assertions follow from the preceding diagonalization and $\Phi_T(\mu) = z_\mu \mathbf{1}_T(\mu)$. Since $z_\mu > 0$, L_T is positive semidefinite. Finally

$$(L_T)_{\alpha\beta} = \langle \Phi_T \chi^\alpha, \chi^\beta \rangle$$

is a tensor-product multiplicity, hence a nonnegative integer. $\square$

Proposition 2.1 is best regarded as a convenient matrix restatement of Schur positivity, not as an independent positivity theorem. Its value here is that it fixes the exact eigenbasis before further multiplication.

2.1. Functoriality under class-set operations. If T_1, T_2 are class subsets with Schur-positive sums, their multiplication matrices are simultaneously diagonal. Hence

$$L_{T_1} L_{T_2} = X Z^{-1} \text{diag}(z_\mu^2 \mathbf{1}_{T_1 \cap T_2}(\mu)) X^\top.$$

This identity is deliberately recorded with the extra factor z_μ : the product of two selectors is not, in general, the selector attached to $T_1 \cap T_2$. Thus the construction is spectral but not Boolean. In particular, nested class sets give nested invariant ranges, yet their matrix products carry centralizer weights. This observation is one reason to keep the $j = 1$ constituent identity separate from higher powers.

3. A CONSTITUENT-LEVEL HIERARCHY

For $\lambda \vdash n$ put $K_\lambda = K(\chi^\lambda)$ and

$$M_T(\lambda) = L_T K_\lambda = K(\Phi_T \chi^\lambda).$$

All multiplication matrices commute and are simultaneously diagonalized by X . Thus $M_T(\lambda)$ is symmetric, integral and entrywise nonnegative, with class eigenvalue

$$z_\mu \chi^\lambda(\mu) \quad (\mu \in T), \quad 0 \quad (\mu \notin T).$$

It need not be positive semidefinite, because $\chi^\lambda(\mu)$ can be negative.

Theorem 3.1 (Constituent-level hierarchy). *Let T be any class subset for which P_T is Schur-positive. For every $j \geq 1$ and $\alpha, \beta, \lambda \vdash n$,*

$$H_{T,j}(\alpha, \beta; \lambda) := \sum_{\mu \in T} z_\mu^{j-1} \chi^\lambda(\mu)^j \chi^\alpha(\mu) \chi^\beta(\mu) = (M_T(\lambda)^j)_{\alpha\beta} \in \mathbb{Z}_{\geq 0}.$$

Proof. Because $\Phi_T \chi^\lambda$ is a genuine character, its j th pointwise power is again a genuine character. The ring homomorphism $f \mapsto K(f)$ gives

$$M_T(\lambda)^j = K((\Phi_T \chi^\lambda)^j),$$

so every entry is a nonnegative integral multiplicity. Alternatively, the eigenvalue on the μ -column is $(z_\mu \chi^\lambda(\mu))^j$ for $\mu \in T$; reconstruction of the (α, β) entry contributes the factor $1/z_\mu$, giving the displayed sum. $\square$

Corollary 3.2 (Unweighted $j = 1$ inequality). *For every Schur-positive T and all $\alpha, \beta, \lambda \vdash n$,*

$$\sum_{\mu \in T} \chi^\lambda(\mu) \chi^\alpha(\mu) \chi^\beta(\mu) \geq 0.$$

The disappearance of centralizer weights is simply the $j = 1$ instance of Theorem 3.1; it is nevertheless useful because it converts Schur positivity of a restricted class sum into an unweighted triple-product inequality retaining two constituent labels.

Proposition 3.3 (Exact rank and kernel). *For every $\lambda \vdash n$,*

$$\text{rank } M_T(\lambda) = \#\{\mu \in T : \chi^\lambda(\mu) \neq 0\}.$$

Its kernel is the span of the character-table columns indexed by $\mu \notin T$ together with those $\mu \in T$ for which $\chi^\lambda(\mu) = 0$.

Proof. This follows immediately from the simultaneous eigenbasis and the eigenvalues $z_\mu \chi^\lambda(\mu)$ on T . $\square$

3.1. Generating functions and recurrences. Fix $T, \alpha, \beta, \lambda$ and form

$$\mathcal{H}_{\alpha\beta}(t) = \sum_{j \geq 1} H_{T,j}(\alpha, \beta; \lambda) t^{j-1}.$$

As a formal power series,

$$\mathcal{H}_{\alpha\beta}(t) = (M_T(\lambda)(I - tM_T(\lambda))^{-1})_{\alpha\beta} = \sum_{\mu \in T} \frac{\chi^\lambda(\mu) \chi^\alpha(\mu) \chi^\beta(\mu)}{1 - t z_\mu \chi^\lambda(\mu)}.$$

The first expression shows that every Taylor coefficient is a nonnegative integer; the second displays the poles and makes the class-by-class spectrum explicit. Consequently each constituent sequence $j \mapsto H_{T,j}(\alpha, \beta; \lambda)$ satisfies the linear recurrence determined by the distinct nonzero numbers $z_\mu \chi^\lambda(\mu)$, $\mu \in T$. This is not an additional positivity theorem, but it records exactly how the infinite hierarchy is controlled by finitely many restricted character values.

Corollary 3.4 (Finite recurrence). *Let $a_1, \dots, a_s$ be the distinct nonzero values among $z_\mu \chi^\lambda(\mu)$ for $\mu \in T$, and write $q(x) = \prod_{i=1}^s (x - a_i) = x^s + q_{s-1}x^{s-1} + \dots + q_0$. Then for every $j \geq 1$,*

$$H_{T,j+s} + q_{s-1}H_{T,j+s-1} + \dots + q_0H_{T,j} = 0,$$

with the constituent labels suppressed from the notation.

Proof. Apply q to $M_T(\lambda)$ on its nonzero spectral subspace and then take the (α, β) entry after multiplication by $M_T(\lambda)^j$. $\square$

3.2. Global selectors force strict constituent positivity. The nonnegativity in Corollary 3.2 can be sharpened whenever the selector character has full irreducible support. This uses information that is lost in the spectral description of L_T .

Write

$$\Phi_T = \sum_{\gamma \vdash n} c_\gamma \chi^\gamma, \quad m_T = \min_{\gamma \vdash n} c_\gamma.$$

We call T *global* if $m_T > 0$, equivalently if Φ_T contains every irreducible character.

Theorem 3.5 (Strict constituent positivity for global selectors). *Suppose P_T is Schur-positive and T is global. Then for every $\lambda, \alpha, \beta \vdash n$,*

$$\sum_{\mu \in T} \chi^\lambda(\mu) \chi^\alpha(\mu) \chi^\beta(\mu) = \langle \Phi_T, \chi^\lambda \chi^\alpha \chi^\beta \rangle \geq m_T > 0.$$

More generally, if Θ is any nonzero genuine character of S_n , then

$$\langle \Phi_T, \Theta \rangle \geq m_T.$$

Proof. Write $\Theta = \sum_\gamma d_\gamma \chi^\gamma$ with $d_\gamma \in \mathbb{Z}_{\geq 0}$ and not all zero. Then

$$\langle \Phi_T, \Theta \rangle = \sum_\gamma c_\gamma d_\gamma \geq m_T \sum_\gamma d_\gamma \geq m_T.$$

Taking $\Theta = \chi^\lambda \chi^\alpha \chi^\beta$, which is a nonzero genuine character, gives the first assertion. The equality with the unweighted class sum follows from $\Phi_T(\mu) = z_\mu$ on T and zero off T . $\square$

This theorem is elementary once globality is known, but it changes the qualitative conclusion of the constituent hierarchy: the $j = 1$ matrices have no zero entries. It also isolates a useful invariant, m_T , whose size controls all triple products simultaneously.

4. THE EVEN NON-SPLIT SET AND PARTITION CONJUGATION

Let D be the set of partitions of n into distinct odd parts; these index the S_n conjugacy classes that split in A_n . Set

$$B = \{\mu \vdash n : (-1)^{n-\ell(\mu)} = 1, \mu \notin D\}.$$

Thus B is the set of even non-split cycle types. Sundaram [5, Theorem 1.1(3)] proves that

$$P_B = \sum_{\mu \in B} p_\mu$$

is Schur-positive. If $p(n)$ is the number of partitions and $q(n) = |D|$, the standard parity count gives

$$|B| = \frac{p(n) - q(n)}{2} =: r_n.$$

Therefore Proposition 2.1 gives a canonical rank- r_n matrix L_B with spectrum z_μ on B and 0 elsewhere, while Theorem 3.1 gives

$$\sum_{\mu \in B} z_\mu^{j-1} \chi^\lambda(\mu)^j \chi^\alpha(\mu) \chi^\beta(\mu) \geq 0.$$

In particular, Corollary 3.2 is completely unweighted on B .

Sundaram proves more than Schur positivity here: the restricted row sums for the even non-split set are strictly positive, so Φ_B contains every irreducible. Theorem 3.5 therefore gives the following strengthening.

Corollary 4.1 (Strict balanced triple-product theorem). *For $n \geq 2$ and every $\lambda, \alpha, \beta \vdash n$,*

$$\sum_{\mu \in B} \chi^\lambda(\mu) \chi^\alpha(\mu) \chi^\beta(\mu) > 0.$$

In fact the left-hand side is at least

$$m_B := \min_{\gamma \vdash n} \sum_{\mu \in B} \chi^\gamma(\mu).$$

The exact computations in degrees $n \leq 13$ suggest that this universal lower bound has a surprisingly simple closed form.

Conjecture 4.2 (Sharp balanced lower bound). *Let B be the even non-split cycle types of S_n . Then for every $n \geq 2$,*

$$m_B = |B| = \frac{p(n) - q(n)}{2},$$

where $q(n)$ is the number of distinct-odd partitions of n . Equivalently,

$$\sum_{\mu \in B} \chi^\gamma(\mu) \geq |B| \quad \text{for every } \gamma \vdash n.$$

Consequently, if the conjecture holds, then every constituent-level triple sum satisfies

$$\sum_{\mu \in B} \chi^\lambda(\mu) \chi^\alpha(\mu) \chi^\beta(\mu) \geq |B|.$$

The trivial and sign characters show that the proposed constant $|B|$ is best possible, since both take value 1 on every even class. Exact Murnaghan–Nakayama computation verifies Conjecture 4.2 for every $2 \leq n \leq 13$. In degrees $n \geq 5$ within this range, equality for the one-row and one-column characters is the only observed minimum; small degrees have additional accidental equalities.

Let V have basis e_α , $\alpha \vdash n$, and define $Je_\alpha = e_{\alpha'}$. Since every $\mu \in B$ is even,

$$\chi^{\alpha'}(\mu) = \chi^\alpha(\mu).$$

Hence, for every $j \geq 1$,

$$H_{B,j}(\alpha', \beta; \lambda) = H_{B,j}(\alpha, \beta; \lambda) = H_{B,j}(\alpha, \beta'; \lambda).$$

Equivalently,

$$JM_B(\lambda)^j = M_B(\lambda)^j = M_B(\lambda)^j J.$$

For $j = 1$, if $V = V_+ \oplus V_-$ is the ± 1 eigenspace decomposition of J , then

$$M_B(\lambda)V_- = 0, \quad \text{im } M_B(\lambda) \subseteq V_+.$$

Thus the localized multiplication matrix factors through the quotient identifying α with α' .

An explicit S_3 example. For $n = 3$, the only even non-split cycle type is the identity class $B = \{(1^3)\}$: (3) is even but distinct-odd, while $(2, 1)$ is odd. Since $z_{(1^3)} = 6$, Φ_B is the regular character. Order the irreducibles as $(3), (2, 1), (1^3)$, with dimension vector

$$d = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

Then

$$L_B = dd^\top = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix},$$

which has rank $1 = |B|$ and nonzero eigenvalue 6. For $\lambda = (2, 1)$,

$$M_B((2, 1)) = 2L_B = \begin{pmatrix} 2 & 4 & 2 \\ 4 & 8 & 4 \\ 2 & 4 & 2 \end{pmatrix}.$$

At $j = 1$,

$$H_{B,1}(\alpha, \beta; (2, 1)) = \chi^{(2,1)}(1^3) \chi^\alpha(1^3) \chi^\beta(1^3) = 2d_\alpha d_\beta,$$

recovering the matrix entry by entry. The first and third rows coincide, so $e_{(3)} - e_{(1^3)} \in V_-$ is annihilated.

4.1. A genuinely two-class S_4 example. The S_3 calculation above is rank one. The first example displaying both the restricted spectrum and the failure of positive semidefiniteness for $M_B(\lambda)$ occurs for S_4 . The even non-split set is

$$B = \{(1^4), (2, 2)\};$$

the remaining even type $(3, 1)$ is distinct-odd and therefore split in A_4 . Order the irreducibles as

$$(4), (3, 1), (2, 2), (2, 1, 1), (1^4).$$

The character-table columns at (1^4) and $(2, 2)$ are respectively

$$d = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 3 \\ 1 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ -1 \\ 2 \\ -1 \\ 1 \end{pmatrix},$$

with centralizer orders 24 and 8. Because the reconstruction formula cancels the factor z_μ for L_B , one obtains

$$L_B = dd^\top + vv^\top = \begin{pmatrix} 2 & 2 & 4 & 2 & 2 \\ 2 & 10 & 4 & 10 & 2 \\ 4 & 4 & 8 & 4 & 4 \\ 2 & 10 & 4 & 10 & 2 \\ 2 & 2 & 4 & 2 & 2 \end{pmatrix}.$$

Its nonzero eigenvalues are 24 and 8, exactly as Proposition 2.1 predicts.

Now choose $\lambda = (3, 1)$. Since $\chi^{(3,1)}(1^4) = 3$ and $\chi^{(3,1)}(2, 2) = -1$,

$$M_B((3, 1)) = 3dd^\top - vv^\top = \begin{pmatrix} 2 & 10 & 4 & 10 & 2 \\ 10 & 26 & 20 & 26 & 10 \\ 4 & 20 & 8 & 20 & 4 \\ 10 & 26 & 20 & 26 & 10 \\ 2 & 10 & 4 & 10 & 2 \end{pmatrix}.$$

Every entry is nonnegative, but the two nonzero eigenvalues are

$$24 \cdot 3 = 72, \quad 8 \cdot (-1) = -8.$$

Thus this single example simultaneously verifies the unweighted $j = 1$ formula, exhibits rank two, and demonstrates concretely why entrywise nonnegativity of the localized Kronecker matrix must not be conflated with positive semidefiniteness. It also makes the J -quotient visible: the rows indexed by (4) and (1^4) coincide, as do those indexed by $(3, 1)$ and $(2, 1, 1)$.

4.2. Compression to conjugate-pair coordinates. Choose one representative from each pair $\{\alpha, \alpha'\}$ with $\alpha \neq \alpha'$, together with the self-conjugate partitions. An orthogonal basis of V_+ is obtained from

$$f_{\{\alpha, \alpha'\}} = \frac{e_\alpha + e_{\alpha'}}{\sqrt{2}}$$

and the vectors e_α for self-conjugate α . In this basis $M_B(\lambda)$ has a single nonzero block $\widetilde{M}_B(\lambda)$ on V_+ and a zero block on V_- . The entries of the compressed block are explicit: for two nonsymmetric conjugate pairs they are twice the corresponding entries of $M_B(\lambda)$; with one self-conjugate index they acquire a factor $\sqrt{2}$; and between two self-conjugate indices they are unchanged. The irrational normalization is only an artifact of choosing an orthonormal basis; using orbit sums instead gives an integral quotient matrix. This makes precise the statement that the B -hierarchy lives on the sign-conjugation quotient rather than merely commuting with it.

Proposition 4.3 (Rank on the quotient). *The nonzero spectrum and rank of the compressed operator on V_+ are exactly those of $M_B(\lambda)$. Hence*

$$\text{rank } \widetilde{M}_B(\lambda) = \#\{\mu \in B : \chi^\lambda(\mu) \neq 0\} \leq r_n.$$

All of V_- lies in the kernel independently of λ .

Proof. The decomposition $V = V_+ \oplus V_-$ is orthogonal, $M_B(\lambda)V_- = 0$, and the image lies in V_+ . The rank statement then follows from Proposition 3.3. $\square$

4.3. A second explicit check: the identity-supported case. Whenever a Schur-positive class set is the singleton $T = \{(1^n)\}$, one has Φ_T equal to the regular character and

$$L_T = dd^\top, \quad d = (f^\alpha)_{\alpha \vdash n},$$

where $f^\alpha = \chi^\alpha(1^n)$. For every λ ,

$$M_T(\lambda) = f^\lambda dd^\top,$$

and therefore

$$H_{T,j}(\alpha, \beta; \lambda) = (n!)^{j-1} (f^\lambda)^j f^\alpha f^\beta.$$

This family provides a useful normalization check on every factor z_μ^{j-1} in Theorem 3.1: the unique centralizer order is $z_{(1^n)} = n!$, and $(dd^\top)^j = (n!)^{j-1} dd^\top$ by $\sum_\alpha (f^\alpha)^2 = n!$.

5. WHAT IS FORMAL, WHAT IS ADDITIONAL, AND WHAT IS NOT IMPLIED

The logical inputs should be kept separate. Character-table diagonalization of representation-ring multiplication is standard. Sundaram supplies the nontrivial Schur-positivity and full-support results for P_B and other restricted class sums [5]. Proposition 2.1 packages these ingredients as an exact PSD multiplication matrix; by itself this is a reformulation. Theorem 3.1 retains matrix entries after multiplication by χ^λ and after taking powers, producing a family of constituent-labelled inequalities. Theorem 3.5 then uses full irreducible support, rather than spectral diagonalization alone, to turn the unweighted $j = 1$ inequalities into strict inequalities. For B this gives Corollary 4.1, while Conjecture 4.2 asks for the sharp universal lower bound suggested by exact computation.

A targeted literature search also located the recent nonnegative-inverse-eigenvalue result of Gershnik–Lewis–Paparella, which proves that finite-group character tables are ideal Perron similarities. That work reinforces the need not to claim character-table diagonalization or nonnegative spectral realization as new here; our strict constituent statement instead uses the full-support property of a particular Schur-positive selector.[4]

Several limitations prevent overinterpretation. First, $M_T(\lambda)$ is generally not PSD; its nonzero eigenvalues are $z_\mu \chi^\lambda(\mu)$ and can have either sign. Second, simultaneous diagonalization means that nesting several class-set selectors cannot by itself create comparison inequalities between

different conjugacy-class sectors. Third, the rank of L_B or $M_B(\lambda)$ does not by itself bound the negative inertia of an unrelated Kronecker multiplication matrix. Any such conclusion requires additional coupling information, not merely the selector decomposition.

The natural next questions are therefore representation-theoretic rather than spectral tautologies: which other Schur-positive class subsets give useful unweighted $j = 1$ inequalities; whether the B -hierarchy admits a direct module or combinatorial interpretation; and what additional structure, beyond simultaneous diagonalization, can couple the invariant quotient to complementary class sectors.

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