EXTREMAL NEGATIVITY IN SYMMETRIC-GROUP CHARACTER TABLES: BINARY COMPRESSION FOR BALANCED HOOKS AND TERMINAL ODD SCALES

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ABSTRACT. Let $N_n^-(\lambda)$ be the number of conjugacy classes of S_n on which the irreducible character χ^λ is negative, with classes counted without size weights. Hopkins' OPAC-038 asks whether the sign representation eventually maximizes this statistic. We isolate the balanced hook $\lambda_k = (k+1, 1^{k-1}) \vdash 2k$ and develop a self-contained binary-compression analysis. Canonical Glaisher split–merge edges $A = \tau \cup (m, m)$, $B = \tau \cup (2m)$ admit an exact two-scale formula in terms of hook coefficients of τ. For odd terminal scales $m > k/3$, every strict negative–negative edge is mapped injectively to a strict positive–positive edge by splitting its unique higher even part. In the first transition band $k/4 < m \leq k/3$, strict negative–negative sources are classified: apart from one explicit double-even family, each has a unique even part. We derive a finite-difference normal form and prove a parity-complete absorption theorem producing favorable all-odd targets from every nonempty residual submultiset of weight below $d = k - 2m$. The current terminal-odd boundary is closed through $d \leq m + 4$ in the project record, while $d = m + 5$ is the first unresolved band: the $b = d - 1$ branch is split-capable in exhaustive tests through odd $m \leq 61$, but the $b = d - 3$ branch already contains split-dead sources. We prove a complete two-part residual classification and an infinite family of split-dead sources, and we record a conditional three-large absorption reservoir. Odd splitting alone, split-capable-only Hall, absorption alone, and a fixed two-small scalar reservoir each fail in explicit examples. Exact integer computation verifies unrestricted hybrid matching through odd $m \leq 41$. The global OPAC-038 problem remains open.

1. Introduction

For partitions $\lambda, \mu \vdash n$, write $\chi^\lambda(\mu)$ for the ordinary irreducible character value of the symmetric group S_n. We study the unweighted sign statistic

$$N_n^-(\lambda) = \#\{\mu \vdash n : \chi^\lambda(\mu) < 0\}.$$

The sign representation has no zero entries and is negative precisely on the odd conjugacy classes. Hopkins' OPAC-038 asks whether

$$N_n^-(\lambda) \leq N_n^-((1^n)) \tag{1}$$

for every $\lambda \vdash n$ once n is sufficiently large [1]. The global problem remains open. The present paper proves a collection of all-degree structural results for one distinguished family, the balanced hooks

$$\lambda_k = (k+1, 1^{k-1}) \vdash 2k.$$

No theorem below is claimed to settle OPAC-038 outside this family, and the finite computations are not used as premises in any proof.

The mechanism is binary compression. Glaisher's split–merge operation $(m, m) \leftrightarrow (2m)$ pairs even and odd conjugacy classes while leaving distinct-odd partitions as the fixed exceptional states. For balanced hooks the exterior-power generating function converts a canonical binary edge into two coefficient scales of the common remainder. This gives an exact local model without introducing class-size weights.

The main proved results are as follows. First, every canonical edge admits the two-scale normal form of Theorem 3.1. Second, for odd terminal scales $m > k/3$, Theorem 3.2 gives an explicit injection from all negative–negative edges to strict positive–positive edges. Third, in the first transition band $k/4 < m \leq k/3$,

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Theorem 4.1 classifies every strict negative–negative source. The only multi-even case is an explicit double-even family; all other sources have one even part. Fourth, the one-even family admits both odd-split targets and a larger absorption reservoir. The parity-complete absorption theorem, Theorem 4.6, treats every nonempty residual submultiset of weight below d. We then incorporate the current boundary record: terminal odd payment is closed through $d \le m+4$, while $d = m+5$ is the first unresolved band. We also prove a complete two-part residual classification and an infinite split-dead family, and record a conditional three-large reservoir. Finally, several natural scalar or one-reservoir Hall strengthenings are false, so the remaining issue is genuinely a collision problem rather than local target existence.

This version is intentionally self-contained. Broader claims from the surrounding OPAC-038 research program are not used here. The only unresolved statement formulated as a conjecture is the hybrid Hall assertion in Section 6; its finite verification is clearly separated from the proved theorems.

2. Parity balance and binary compression

For a partition $\mu \vdash n$, put $\varepsilon(\mu) = (-1)^{n-\ell(\mu)}$. Let E_n and O_n denote the even and odd conjugacy classes. Let $p(n)$ be the partition number and let $q(n)$ be the number of self-conjugate partitions, equivalently the number of partitions into distinct odd parts.

Proposition 2.1 (Sign-row count)**.** *For every $n \ge 0$,*

$$\sum_{\mu \vdash n} \varepsilon(\mu) = q(n), \qquad N_n^-((1^n)) = \frac{p(n)-q(n)}{2}.$$

Proof. The signed class generating function is

$$\sum_{n\ge 0}\left(\sum_{\mu\vdash n}\varepsilon(\mu)\right)x^n = \prod_{j \text{ odd}} \frac{1}{1-x^j} \prod_{j \text{ even}} \frac{1}{1+x^j}.$$

Using $(1+x^{2r})^{-1} = (1-x^{2r})/(1-x^{4r})$ and cancelling the even Euler factors gives $\prod_{r\ge 1}(1+x^{2r-1})$, the generating function for distinct-odd partitions. If the sign row has P_n positive and M_n negative entries, then $P_n + M_n = p(n)$ and $P_n - M_n = q(n)$. □

For a fixed row λ, let $E^-(\lambda)$ be the number of negative values on E_n, and define $O^-(\lambda), O^0(\lambda), O^+(\lambda)$ analogously on O_n.

Proposition 2.2 (Parity balance)**.** *For every $\lambda \vdash n$,*

$$N_n^-(\lambda') - N_n^-((1^n)) = E^-(\lambda) - O^-(\lambda) - O^0(\lambda).$$

Consequently (1) *at degree n is equivalent to*

$$E^-(\lambda) \le O^-(\lambda) + O^0(\lambda) \qquad (\lambda \vdash n). \tag{2}$$

Proof. The sign twist gives $\chi^{\lambda'}(\mu) = \varepsilon(\mu)\chi^\lambda(\mu)$, hence $N_n^-(\lambda') = E^-(\lambda) + O^+(\lambda)$. The sign row is negative on every odd class, so $N_n^-((1^n)) = O^-(\lambda) + O^0(\lambda) + O^+(\lambda)$. Subtract. □

The zero term is essential: in S_3, the row $\chi^{(2,1)}$ has values $2, 0, -1$, so $E^- = 1$, $O^- = 0$, and $O^0 = 1$.

For $0 \le r \le n-1$, write $H_r^{(n)} = \chi^{(n-r,1^r)}$.

Proposition 2.3 (Hook generating function)**.** *For $\mu = (\mu_1, \dots, \mu_\ell) \vdash n$,*

$$\sum_{r=0}^{n-1} H_r^{(n)}(\mu) t^r = \frac{1}{1+t}\prod_{i=1}^{\ell}\left(1-(-t)^{\mu_i}\right). \tag{3}$$

Moreover

$$H_r^{(n)}(\mu) = \varepsilon(\mu) H_{n-1-r}^{(n)}(\mu). \tag{4}$$

Proof. This is the standard exterior-power form of the hook Murnaghan–Nakayama formula; see [2, 4]. Equation (4) follows from conjugation of hook partitions and sign twisting. □

Set

$$X_k(\mu) = \chi^{\lambda_k}(\mu) = H^{(2k)}_{k-1}(\mu).$$

The binary geometry is governed by $(m,m) \leftrightarrow (2m)$. Writing every positive integer uniquely as $2^j b$ with b odd, repeated merging along each odd-base chain gives the usual Glaisher correspondence. Fix a canonical first active chain and level; every non-singleton state is paired by one split–merge edge and the fixed points are exactly the distinct-odd partitions.

Definition 2.4 (Terminal odd edge)**.** Let m be odd. A pair

$$A = \tau \cup (m,m), \qquad B = \tau \cup (2m) \tag{5}$$

is a terminal odd m-edge if every odd-base chain $c < m$ is already Glaisher-trivial in τ and the m-chain of τ contains no part $2^j m$ with $j \geq 1$.

Thus every even part of the common remainder τ has odd base strictly larger than m.

Lemma 2.5 (All-odd middle values)**.** *If $\mu \vdash 2k$ has only odd parts, then $X_k(\mu) \geq 0$.*

Proof. Put $P(t) = \prod_i(1 + t^{\mu_i})$. Since $|\mu| = 2k$ and all parts are odd, $\ell(\mu)$ is even, so $P(-1) = 0$ and $Q(t) = P(t)/(1+t)$ is a polynomial of degree $2k-1$. It is palindromic, hence its coefficients at t^{k-1} and t^k agree. Since $(1+t)Q = P$, twice that common coefficient is $[t^k]P \geq 0$. By (3), the coefficient at t^{k-1} is $X_k(\mu)$. □

On each paired Glaisher edge exactly one endpoint is an even class and one is an odd class. Let N_{--} and N_{++} be the numbers of canonical edges whose two balanced-hook endpoint values are respectively both negative and both nonnegative.

Proposition 2.6 (Binary edge deficit)**.**

$$E^-(\lambda_k) - O^+(\lambda_k) - O^0(\lambda_k) = N_{--} - N_{++}.$$

Hence an injection from the $--$ edges to the $++$ edges proves parity balance on the corresponding collection of canonical edges.

Proof. Distinct-odd fixed states are nonnegative by Lemma 2.5. A paired edge contributes $+1$ to the displayed deficit if both endpoint values are negative, -1 if both are nonnegative, and 0 otherwise. □

3. Exact two-scale formula and the terminal odd theorem

Let (5) be a canonical edge and put

$$q = k - m, \qquad d = k - 2m, \qquad |\tau| = 2q.$$

For a partition ρ, let $H_j(\rho)$ denote the coefficient of t^j in its hook polynomial (3), with $H_j = 0$ outside the natural range.

Theorem 3.1 (Two-scale normal form)**.** *For the edge* (5),

$$X_k(A) = \varepsilon(\tau)H_d(\tau) - 2(-1)^m H_{q-1}(\tau) + H_{d-1}(\tau), \tag{6}$$

$$X_k(B) = \varepsilon(\tau)H_d(\tau) - H_{d-1}(\tau). \tag{7}$$

Proof. From (3),

$$F_A(t) = F_\tau(t)(1-(-t)^m)^2, \qquad F_B(t) = F_\tau(t)(1-t^{2m}).$$

Extracting t^{k-1} gives

$$X_k(A) = H_{k-1}(\tau) - 2(-1)^m H_{q-1}(\tau) + H_{d-1}(\tau),$$
$$X_k(B) = H_{k-1}(\tau) - H_{d-1}(\tau).$$

Since $|\tau| = 2q$, the complementary index to $k-1$ in degree $2q-1$ is d. Apply (4). □

For odd m, the middle term in (6) is $+2H_{q-1}(\tau)$.

Theorem 3.2 (Terminal odd one-third theorem)**.** *Let m be odd and let* (5) *be a terminal odd m-edge. If $m > d = k - 2m$, equivalently $m > k/3$, then every $--$ edge is mapped injectively to a strict $++$ edge at the same m-scale. Consequently all terminal odd m-edges with $m > k/3$ contribute nonpositively to $N_{--} - N_{++}$.*

Proof. Since $|\tau| = 2(m+d) < 4m$, every even part of a terminal odd remainder exceeds $2m$, so τ contains at most one even part; a level-≥ 2 even part would exceed $4m$. If τ has no even part, it is all odd and Lemma 2.5 prevents a $--$ source. Hence

$$\tau^- = \sigma \cup (2b),$$

where $b > m$ is odd and σ is all odd. Write $|\sigma| = 2a$, so $a + b = q = m + d$. Put

$$F(t) = \frac{\prod_{x\in\sigma}(1+t^x)}{1+t} = \sum_j f_j t^j.$$

Because $b > m > d$, one has $a = q - b < d < m$. Set $s = 2a - d = k - 2b$. Palindromicity of F gives $f_d = f_{s-1}$ and $f_{d-1} = f_s$. Since $2b > d$, Theorem 3.1 reduces to

$$(X_k(A^-), X_k(B^-)) = (f_s - f_{s-1}, -(f_{s-1} + f_s)).$$

For a $--$ source define $D = f_{s-1} - f_s > 0$ and $P = f_{s-1} + f_s > 0$, so the pair is $(-D, -P)$.

Now split the unique higher even part and put $\tau^+ = \sigma \cup (b, b)$. This is all odd and remains canonical at the same terminal scale. The low-degree endpoint difference is $X_k(B^+) = D$. Moreover $H_{q-1}(\tau^+) = 2f_{a-1}$. Since F is palindromic of degree $2a - 1$, $f_{a-1} = f_a =: C$, and

$$2C = [t^a] \prod_{x\in\sigma}(1 + t^x) \geq 0.$$

Therefore

$$(X_k(A^+), X_k(B^+)) = (P + 4C, D),$$

which is strictly positive in both coordinates. The map is injective because the repeated higher odd part b and then σ are recovered from τ^+. □

4. The first transition band and parity-complete absorption

Assume from now on that

$$m \leq d < 2m, \qquad \text{equivalently} \qquad \frac{k}{4} < m \leq \frac{k}{3},$$

with m odd. Then $|\tau| = 2(m + d) < 6m$.

Theorem 4.1 (First-band even-part classification)**.** *Let* (5) *be a terminal odd m-edge with $m \leq d < 2m$. If both endpoint values are negative, then exactly one of the following holds:*

(a) $\tau = \sigma \cup (2b)$, where $b > m$ is odd and σ is a nonempty all-odd multiset;

(b) $\tau = (2b, 2b)$, necessarily $q = 2b$, and the endpoint pair is $(-2, -2)$.

No other multi-even or higher-level even configuration can be $--$.

Proof. Every even part of τ has odd base $> m$, hence is $> 2m > d$. Suppose τ contains at least two even parts and at least one odd part. If W is the total weight of the odd parts, then the even parts contribute $> 4m$, so

$$W < 2(m + d) - 4m = 2(d - m) < d.$$

The odd quotient polynomial has degree $W - 1 < d - 1$, while every even factor is $1 - t^e$ with $e > d$. Hence $H_{d-1}(\tau) = H_d(\tau) = 0$, and (7) gives $X_k(B) = 0$, contradicting strict negativity.

If there are at least two even parts and no odd part, size forces exactly two. Neither can have binary level at least 2, so $\tau = (2b, 2c)$ with $b, c > m$ odd. For $r \leq d < 2m < 2b, 2c$, the relevant coefficients are those of $(1 + t)^{-1}$: $H_d = (-1)^d$ and $H_{d-1} = (-1)^{d-1}$. Thus $X_k(B) = 2(-1)^d$, so $B < 0$ forces d odd. Then $q = m + d$ is even and $X_k(A) = 2H_{q-1}(\tau)$. The double product lies above degree $q - 1$, and a direct expansion gives $H_{q-1} = N - 1$, where N is the number of parts among $2b, 2c$ not exceeding $q - 1$. Negativity forces $N = 0$. Since $2b + 2c = 2q$, this implies $2b = 2c = q$, giving case (b) and endpoint pair $(-2, -2)$.

Finally, suppose there is exactly one even part. If its binary level is at least 2, it exceeds $4m$. With odd parts present their total weight is $< 2(d - m) < d$, again giving $H_{d-1} = H_d = 0$ and $B = 0$; without odd parts direct substitution also gives $B = 0$. Thus any strict one-even source has the form in (a), and σ is nonempty. □

Proposition 4.2 (Pure double-even payment). *In case (b), the map*

$$(2b, 2b) \longmapsto (2b, b, b)$$

produces a canonical target with endpoint pair $(0, 0)$*, injectively in* b*.*

Proof. Here $q = 2b$ and d is odd. For $\tau^+ = (2b, b, b)$, the factor $1 - t^{2b}$ lies above degrees $d, d-1$, while $(1+t^b)^2/(1+t)$ has $H_d = -(-1)^d$ and $H_{d-1} = (-1)^d$. Thus (7) gives $B = 0$. Also $H_{q-1} = 1$, and (6) gives $A = 2 + 2(-1)^d = 0$. □

It remains to analyze $\tau^- = \sigma \cup (2b)$. Write $|\sigma| = 2a$, $a + b = q = m + d$, and

$$F(t) = \frac{\prod_{x\in\sigma}(1+t^x)}{1+t} = \sum_j f_j t^j. \tag{8}$$

Set $u = a - b = q - 2b$ and $s = 2a - d = m + u = k - 2b$.

Proposition 4.3 (One-even coefficient normal form). *For* $\tau^- = \sigma \cup (2b)$,

$$X_k(B^-) = -(f_{s-1} + f_s) = -[t^s] \prod_{x\in\sigma}(1+t^x), \tag{9}$$

$$X_k(A^-) = (f_s - f_{s-1}) + 2(f_u - f_{u-1}). \tag{10}$$

Proof. The polynomial F has degree $2a - 1$ and is palindromic, so $f_d = f_{s-1}$, $f_{d-1} = f_s$, $f_{q-1} = f_u$, and $f_{q-1-2b} = f_{u-1}$. The unique even part gives $\varepsilon(\tau^-) = -1$ and contributes $1 - t^{2b}$. Substitute into Theorem 3.1; the second equality in (9) follows from $(1+t)F = \prod_{x\in\sigma}(1+t^x)$. □

An odd split $2b = v + w$ with v, w odd gives $\tau_{v,w} = \sigma \cup (v, w)$. Put $\Delta f_j = f_j - f_{j-1}$.

Proposition 4.4 (Odd-split criterion). *For every canonical odd split* $2b = v + w$*, one target endpoint is automatically nonnegative. The other is*

$$B_{v,w} = \Delta f_d + \Delta f_{d-v} + \Delta f_{d-w}.$$

Thus $B_{v,w} \ge 0$ *is sufficient for a ++ target.*

Proof. Expand $F(t)(1+t^v)(1+t^w)$. Since $2b > d$, the f_{d-2b} term vanishes, and $H_d - H_{d-1}$ is the displayed finite difference. The other endpoint is $(H_d + H_{d-1}) + 2H_{q-1}$; for an all-odd partition both terms are nonnegative by subset counting and Lemma 2.5. □

Odd splitting is not universal.

Example 4.5 (All odd splits can fail). Take $m = 11$, $d = 21$, $k = 43$, $b = 13$, and $\sigma = (17, 15, 5, 1)$. Then $\tau^- = (26, 17, 15, 5, 1)$ has endpoint pair $(-1, -1)$. The canonical odd splits of 26 give

$$(3, 23) : (4, -2), \quad (7, 19) : (3, -1), \quad (9, 17) : (5, -1),$$

$$(11, 15) : (6, -2), \qquad (13, 13) : (3, -1).$$

Hence no canonical odd split is a ++ target.

The complementary construction is absorption. The following form treats both parities of the residual submultiset.

Theorem 4.6 (Parity-complete absorption). *Let* $\tau^- = \sigma \cup (2b)$ *be a strict one-even* $--$ *source in the first band, and let* $\varnothing \ne R \subseteq \sigma$ *be any submultiset with* $w = \mathrm{wt}(R) < d$*.*

(i) If $|R|$ *is odd, then*

$$Z_R = R \cup (2q - w)$$

is all odd and has endpoint pair $(0, 0)$*.*

(ii) If $|R|$ *is even and* d *is even, set* $L = 2q - w$ *and*

$$Z_R = R \cup (d + 1, L - d - 1).$$

Both new parts are odd and exceed d*, and the endpoint pair is* $(2H_{q-1}(Z_R), 0)$*.*

(iii) If $|R|$ is even and d is odd, set $L = 2q - w$ and

$$Z_R = R \cup (d, L - d).$$

Then the endpoint pair is $(1 + 2H_{q-1}(Z_R), 1)$.

In every case the target is favorable. In particular, every strict one-even source has an explicit favorable absorption target.

Proof. Since $w < d$ and $q = m + d$,

$$L = 2q - w > 2q - d = 2m + d > 2d.$$

If $|R|$ is odd, then w and L are odd. The quotient $Q_R(t) = \prod_{x\in R}(1 + t^x)/(1 + t)$ has degree $w - 1 < d - 1$, while the giant part $L > k$ lies above $d - 1, d, q - 1$. Hence $H_{d-1} = H_d = H_{q-1} = 0$, giving $(0, 0)$ by Theorem 3.1.

Now suppose $|R|$ is even. Then w and L are even. If d is even, $d + 1$ and $L - d - 1$ are odd; because $L > 2d$, both exceed d. Thus $H_{d-1} = H_d = 0$, and the endpoints are $(2H_{q-1}, 0)$. Since the target is all odd of size $2q$, Lemma 2.5 gives $H_{q-1} \geq 0$.

If d is odd, both d and $L - d$ are odd and $L - d > d$. The residual quotient has degree $< d - 1$, so multiplication by $(1 + t^d)(1 + t^{L-d})$ gives $H_{d-1} = 0$ and $H_d = 1$. The target is all odd, hence $\varepsilon = 1$, and Theorem 3.1 gives $(1 + 2H_{q-1}, 1)$ with $H_{q-1} \geq 0$.

Finally, a strict one-even source has nonempty σ and $\mathrm{wt}(\sigma) = 2a < 2d$ because $b > m$ and $a + b = m + d$. Therefore some singleton $R = (x)$ satisfies $x < d$, so case (i) always supplies at least one target. □

Remark 4.7. The parity-complete theorem enlarges the absorption reservoir but does not solve collision control: distinct sources can still produce the same target. This distinction between local existence and global capacity is essential.

5. Boundary extension and calibrated obstructions

The preceding first-band formulas permit a sharper description near the upper terminal boundary. We separate proved manuscript results from the strongest current project-record extension whenever the compact proof is not reproduced here.

Proposition 5.1 (Equal-split boundary theorem)**.** *In the first band, every one-even bad source with $b \geq d$ is paid injectively by the equal split $2b \mapsto b + b$. Consequently every terminal odd band with*

$$d \leq m + 2, \qquad d < 2m,$$

is closed.

Proof. When $b \geq d$, one has $a = m + d - b \leq m$ and $u = a - b < 0$. Palindromicity of F, of degree $2a - 1$, gives $\Delta f_d = -\Delta f_s$. Since $u < 0$, the source A-endpoint is $\Delta f_s < 0$. For the equal split, Proposition 4.4 gives

$$B^+ = \Delta f_d + 2\Delta f_{d-b} = -X_k(A^-) + 2\mathbf{1}_{b=d} > 0.$$

The other endpoint is nonnegative for every all-odd target, being a sum of subset coefficients and the nonnegative middle coefficient. The repeated higher odd part $b > m$ recovers the source, proving injectivity. For $d = m, m + 1, m + 2$, every permitted one-even level satisfies $b \geq d$; the pure-double family is handled by Proposition 4.2. □

Remark 5.2 (Current all-degree boundary extension)*.* The current verified research record extends Proposition 5.1 through the complete boundary

$$\boxed{d \leq m + 4, \qquad d < 2m.}$$

For $d = m + 3$, the sole new one-even level is $b = d - 1$, where $u = -1$ and $s = d - 4$; a residual-dependent complement split gives a canonical favorable target and an injective payment disjoint from the higher-b equal-split images. For $d = m + 4$, the sole new level is $b = d - 2$, and the ordinary equal split is favorable because source negativity forces $-\Delta f_m \geq 3$ while the correction $2\Delta f_2$ is nonnegative. Pure double-even sources are paid by Proposition 4.2. An independent exact boundary checker recomputes the ambient hook endpoints and injectivity for every applicable odd $m \leq 51$, covering 26,413 one-even bad sources (22,599 equal-split images and 3,814 complement-split images) plus 24 pure-double payments. The compact complement-split derivation is part of the current project record but is not reproduced in full in this manuscript; accordingly, this paragraph records the latest boundary status without using the $d = m + 3, m + 4$ extension as a premise later in the paper.

The next band already shows why a single deterministic split rule is unlikely to suffice.

Proposition 5.3 (The first unresolved boundary)**.** *At $d = m + 5$ there are two new one-even sublevels below d, namely $b = d - 1$ and $b = d - 3$. The $b = d - 1$ branch is split-capable for all 11,963 bad sources found by exact enumeration through odd $m \leq 61$, but the proposed universal three-smallest-candidate proof is false. At*

$$(m, d, b, \sigma) = (13, 18, 17, (11, 9, 7, 1)),$$

the source has endpoints $(-1, -1)$; the candidate based on $c = 1$ gives $(3, -1)$, while the $c = 7$ and $c = 9$ complement choices are noncanonical, although favorable splits do exist at

$$(3, 31),\ (5, 29),\ (13, 21),\ (15, 19).$$

For the $b = d - 3 = m + 2$ branch, parameterizing a split by $v = d - r$ gives

$$B_r = -\Delta f_{d-4} + \Delta f_r + \Delta f_{6-r},$$

and hence, for odd $r \geq 7$,

$$B_r = -\Delta f_{d-4} + \Delta f_r.$$

This branch contains split-dead sources: at $(m, d, b) = (13, 18, 15)$ with $\sigma = (17, 9, 5, 1)$ the source endpoints are $(-1, -1)$ and no canonical two-part odd split is favorable. Thus $d = m+5$ is the first currently unresolved boundary band.

The absorption theorem admits a useful conditional enlargement.

Proposition 5.4 (Three-large reservoir)**.** *Let $R \subseteq \sigma$ have odd cardinality and weight $w < 2m - d$. If odd integers $L_1, L_2, L_3 > d$ satisfy*

$$L_1 + L_2 + L_3 = 2q - w,$$

then

$$R \cup (L_1, L_2, L_3)$$

is an all-odd favorable target.

Proof. The hypothesis $w < 2m - d \leq d$ implies that the quotient polynomial contributed by R has degree $w - 1 < d - 1$. Since all three added parts exceed d, one has $H_{d-1} = H_d = 0$. The remaining middle coefficient is nonnegative by Lemma 2.5, so Theorem 3.1 gives a nonnegative endpoint pair. ☐

A particularly rigid subfamily can be classified completely.

Theorem 5.5 (Two-part residual classification)**.** *Suppose $\sigma = (x, y)$ with $x \leq y$ odd in the first band. Then $\sigma \cup (2b)$ is bad if and only if*

$$x < m, \qquad y = d, \qquad d - x \in 4\mathbb{Z}_{>0}, \qquad b = m + \frac{d - x}{2}.$$

In this case the source endpoints are $(-1, -1)$ and the equal split $2b \mapsto b + b$ is an injective strict favorable payment.

Proof. Here $a = (x + y)/2 < d$ and $s = x + y - d < a$. A subset sum of (x, y) in this range can only be 0 or x. The case $s = 0$ gives a positive A-endpoint, so badness forces $s = x$, hence $y = d$. In

$$F(t) = \frac{(1 + t^x)(1 + t^y)}{1 + t},$$

the coefficients alternate below x, vanish from x through $y - 1$, and satisfy $f_y = 1$. Thus $\Delta f_x = -1$. If $x < m$, then $u = x - m < 0$ and the source A-endpoint is -1; the cases $x = m$ and $x > m$ give positive A-endpoint. The parity condition that b be odd is exactly $4 \mid (d - x)$. For the equal-split target,

$$B^+ = 1 + 2\Delta f_{d-b},$$

and $d-b$ is even and strictly below x because $d<2m$, so $B^+\in\{1,3,5\}$. The other endpoint is strictly positive by the all-odd subset-coefficient expression. The repeated higher odd value b recovers the source, proving injectivity. □

Finally, split-dead behavior is not sporadic.

Theorem 5.6 (Infinite split-dead family)**.** *Let a be odd, $j\geq 1$, and $a\geq 4j+1$. Set*

$$m=2a+1,\quad d=3a+4j+2,\quad b=2a+2j+1,$$

$$\sigma=(1,a,2a+4j+1,3a+2).$$

Then $m\leq d<2m$, the source $\sigma\cup(2b)$ is canonical with endpoints $(-1,-1)$, and every canonical two-part odd split of $2b$ has strictly negative B-endpoint.

Proof. Put $B_0=2a+4j+1$ and $C_0=3a+2$. Then $d=a+B_0+1$, $u=a+1$, and $s=C_0$. The part 1 cancels the denominator in (8), leaving

$$F=(1+t^a)(1+t^{B_0})(1+t^{C_0}).$$

Thus $(f_s,f_{s-1})=(1,0)$ and $(f_u,f_{u-1})=(0,1)$, giving source endpoints $(-1,-1)$. Since d is odd, the correction indices $d-v,d-w$ are even. Among even degrees below d, F has positive coefficients only at 0 and $a+B_0=d-1$, while $\Delta f_d=-1$. A nonnegative split B-endpoint would therefore force a split part to equal 1 or d. The former repeats the existing part 1; the latter has companion $2b-d=a$, repeating the existing part a. Both violate canonicity. □

The smallest member is $(m,d)=(11,21)$, recovering Example 4.5. Each member nevertheless has four singleton zero-absorption targets from Theorem 4.6. This sharply separates split failure from failure of the hybrid architecture.

6. The hybrid Hall frontier

For strict one-even sources, let T_{split} be the favorable all-odd targets obtained from canonical odd splits satisfying Proposition 4.4, and let T_{abs} be the favorable absorption targets from Theorem 4.6, optionally enlarged by Proposition 5.4. The pure double-even targets from Proposition 4.2 form a third, disjoint family containing an even part.

The original odd-cardinality absorption targets have a giant part $>k$, whereas odd-split targets have all parts below k in the strict one-even regime, so these two basic target spaces are disjoint. Nevertheless, local abundance does not imply global capacity.

Proposition 6.1 (Absorption-only Hall failure)**.** *At $(m,d)=(11,20)$ there are 27 one-even $--$ sources, but the union of their original zero-absorption targets contains only 24 distinct partitions. Hence the zero-absorption reservoir alone cannot saturate that band.*

Proof. This is an exact finite enumeration using integer arithmetic. It is recorded as a computational proposition rather than used as an input to any symbolic theorem. □

Proposition 6.2 (Split-capable-only Hall failure)**.** *At $(m,d)=(27,53)$ the two sources*

$$\tau_1=(58,37,21,15,13,9,7),\qquad \tau_2=(62,37,17,15,13,9,7)$$

have endpoint pairs $(-1,-1)$ and $(-7,-1)$, respectively. Each is split-capable, but the complete favorable split neighborhood of each consists of the same single target

$$T=(41,37,21,17,15,13,9,7),$$

whose endpoint pair is $(6,0)$. Thus the split-capable subgraph itself violates Hall's condition.

Proof. For τ_1 the unique favorable split is $58=17+41$; for τ_2 it is $62=21+41$. Exact evaluation of Proposition 4.4 over every canonical odd split gives no other nonnegative B-endpoint. Hence the two-source set has a one-element neighborhood. □

A further scalar compression is also impossible. Retaining only the two smallest residual parts and completing with two large parts gives favorable targets sourcewise, but at $(m, d) = (35, 69)$ there are 17 split-dead sources with the same two-small fingerprint $\{1, 3\}$ while that restricted reservoir has capacity only 16. Thus the full residual-submultiset system of Theorem 4.6 cannot be replaced by this fixed scalar fingerprint.

These counterexamples rule out odd-split-only Hall, absorption-only Hall, split-capable-only Hall, and the fixed two-small scalar-capacity rule. The surviving collision problem is genuinely hybrid.

Conjecture 6.3 (Unrestricted first-band hybrid Hall conjecture). *Fix odd m and $m \le d < 2m$. Form the bipartite graph from all strict one-even $--$ sources to the union of their favorable canonical odd-split targets and favorable absorption targets. Then this combined graph has a matching saturating the source side.*

Proposition 6.4 (Current finite verification). *Exact maximum matching verifies Conjecture 6.3 for every odd $m \le 41$ and every $m \le d < 2m$. Across the tested bands there are* 1,983,369 *bad one-even sources in* 435 *nonempty bands, including* 2,319 *split-dead sources. Every unrestricted hybrid graph is saturated. The largest tested band is $(m, d) = (41, 81)$, with* 79,648 *sources and* 157 *split-dead sources. Five tested bands have split-only deficiencies.*

The finite verification is evidence, not proof. The optimized matching implementation was exhaustively cross-checked against direct hook-polynomial source and split enumeration on 14,837 sources in 144 bands through $m = 23$, and selected larger bands were independently audited by direct endpoint and target calculations. None of these checks is used as a premise in the symbolic theorems above.

7. Even scales, reproducibility, and limitations

The preceding theorems concern terminal odd scales. At even 2-adic scales, individual-scale positivity is false in exact examples, so the correct algebraic unit is larger than one edge. One exact identity explains the expected aggregation.

Let $X_r(\mu) = H_r^{(|\mu|)}(\mu)$ for a general hook character.

Lemma 7.1 (One-scale Hadamard descent). *For every partition ρ and positive integer a,*

$$X_r(\rho \cup (a, a)) - X_r(\rho \cup (2a)) = 2(-1)^{a+1} X_{r-a}(\rho \cup a).$$

Proof. In the hook generating function put $s = (-t)^a$. The difference of the two edge factors is $(1 - s)^2 - (1 - s^2) = -2s(1 - s)$. Multiply by the generating function of ρ and extract t^r. □

Theorem 7.2 (Binary-square descent). *For all positive integers a, b,*

$$\begin{aligned} &X_r(\rho \cup (a, a, b, b)) - X_r(\rho \cup (2a, b, b)) \\ &\qquad - X_r(\rho \cup (a, a, 2b)) + X_r(\rho \cup (2a, 2b)) = 4(-1)^{a+b} X_{r-a-b}(\rho \cup a \cup b). \end{aligned}$$

Proof. Apply Lemma 7.1 first in the a direction and then in the b direction. □

Theorem 7.2 is deliberately not advertised as a positivity theorem. It shows only that cancellation at even scales naturally lives on a two-dimensional binary cell. A future argument must convert such identities into an unweighted deficit or capacity statement.

Exact computation. The computations use exact integer arithmetic. The original first-band checker `opac_first_band_hybrid_check.py` has SHA-256

`49a70d3129f6c9f87d726d01f60ab50c16a174d975c2261b8dfacc800a80acf3`.

It independently calibrates the structural classification and the source, split, and zero-target formulas. The later boundary checker `opac_boundary_extension_check.py` has SHA-256

`e27af55d4052d767b18b13b186bc3508b5b247cf7ec34752ab0a0bac36efa331`,

and checks 26,413 one-even bad sources through odd $m \le 51$ plus 24 pure-double payments. The dedicated $d = m + 5$, $b = d - 1$ checker `opac_dm5_bdm1_split_check.py` has SHA-256

`43ba4931a67cea534c2b1ead5de68016ebef31c5d5179fe24ddbaa9c9ebbaffd` and finds all 11,963 bad sources split-capable through odd $m \leq 61$. The parity-complete absorption checker `opac_parity_complete_absorption_check.py` has SHA-256

`a56317f979ec068ffa61105a42f7d46b1faae6ae05eb014753dd1380a976c62a`

and audits 8,924 targets from 828 bad sources through odd $m \leq 15$. These calculations calibrate, verify implementations, and falsify stronger conjectures; they are not proof steps for the symbolic theorems.

Limitations and current frontier. Three limitations are explicit. First, the paper treats balanced hooks, not arbitrary partitions. Second, even within terminal odd scales the first transition band still requires the global collision statement in Conjecture 6.3; the first unresolved boundary is already visible at $d = m + 5$, where one sublevel is only CHECK-supported and another contains split-dead sources. Third, the even-scale identity does not yet control signs. These are mathematical frontiers rather than hidden assumptions. In particular, neither the boundary computations nor the hybrid matching computations imply a global OPAC-038 theorem.

8. Conclusion

Binary compression gives a sharp local description of balanced-hook negativity. Above the one-third terminal odd scale, every negative–negative edge has a unique higher even part and one further split reverses both endpoint signs injectively. Immediately below one third, the geometry remains rigid enough to classify every strict source, isolate the double-even exception, and reduce all remaining sources to a one-even finite-difference model. The parity-complete absorption theorem shows that favorable targets are abundant at the local level: every nonempty residual submultiset of weight below d yields a favorable all-odd target, with the endpoint pair determined explicitly by the parity of the residual and of d.

What remains is global. The current boundary record closes terminal odd payment through $d \leq m + 4$, but $d = m + 5$ already separates a strongly split-capable branch from a genuinely split-dead branch. The infinite split-dead family shows that this phenomenon persists far inside the first band. Absorption supplies broad local capacity, including parity-complete and three-large reservoirs, yet scalar compressions and one-reservoir Hall statements fail by collisions. Even scales, meanwhile, require aggregation over binary cells or longer chains. Thus the balanced-hook program has reached two precise interfaces: an unrestricted hybrid Hall theorem for terminal odd scales and a capacitated binary-square theorem at even scales. Neither is assumed here, and OPAC-038 remains open.

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