

SPLIT FUSION IN THE ALTERNATING GROUP AND MULTIPLICITY BOUNDS FOR PRINCIPAL-HOOK CHARACTER VALUES

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ABSTRACT. We first isolate an index-two split-fusion principle for a normal subgroup $H \triangleleft G$ of index two: whenever a family of split restrictions has orthogonal difference characters Δ_i that are eigenvectors for multiplication by a restricted irreducible character, the two fusion channels are exactly the half-sum and half-difference of the ambient tensor multiplicity and the corresponding difference-sector eigenvalue. Specializing to S_n/A_n , let $\gamma \vdash n$ be self-conjugate, let $\delta(\gamma)$ be its distinct-odd principal-hook partition, and write $\text{Res}_{A_n}^{S_n} \chi^\gamma = \alpha_\gamma + \beta_\gamma$. For a non-self-conjugate partition λ , set $\psi_\lambda = \text{Res}_{A_n}^{S_n} \chi^\lambda$ and $\Delta_\gamma = \alpha_\gamma - \beta_\gamma$. We prove that Δ_γ is an exact eigenvector for multiplication by ψ_λ , with eigenvalue $\chi^\lambda(\delta(\gamma))$. Resolving the two fusion channels between split constituents gives

$$a_{ij} = \frac{g(\lambda, \gamma_i, \gamma_j) + \chi^\lambda(\delta_i)\delta_{ij}}{2}, \quad b_{ij} = \frac{g(\lambda, \gamma_i, \gamma_j) - \chi^\lambda(\delta_i)\delta_{ij}}{2}.$$

Consequently $g(\lambda, \gamma_i, \gamma_i) \geq |\chi^\lambda(\delta_i)|$ with the corresponding parity congruence, while off-diagonal coefficients $g(\lambda, \gamma_i, \gamma_j)$ are even. We identify the exact defect from equality as twice the smaller diagonal fusion multiplicity, so equality holds precisely when one diagonal channel vanishes. Moreover the self-conjugate Kronecker block satisfies $G_\lambda \equiv X_\lambda \pmod{2}$, which determines its rank, characteristic polynomial, and all principal minors modulo 2 from the principal-hook values. In matrix form the split-fusion operator decomposes as $G_\lambda \oplus X_\lambda$, where G_λ is the self-conjugate Kronecker block and X_λ is diagonal with entries $\chi^\lambda(\delta_i)$. The A_n splitting mechanism already underlies the principal-hook nonvanishing criterion of Pak–Panova–Vallejo; the refinement here is to retain both split fusion channels simultaneously, yielding exact multiplicity identities, magnitude bounds, and parity constraints.

1. INTRODUCTION

For partitions $\lambda, \mu, \nu \vdash n$, the Kronecker coefficient

$$g(\lambda, \mu, \nu) = \langle \chi^\lambda \chi^\mu, \chi^\nu \rangle_{S_n}$$

is the multiplicity of χ^ν in the tensor product $\chi^\lambda \otimes \chi^\mu$. In contrast with Littlewood–Richardson coefficients, no general positive combinatorial rule is known, and even positivity is difficult. Character-theoretic criteria are therefore valuable when they convert a multiplicity question into a tractable evaluation problem.

A particularly useful criterion occurs for self-conjugate partitions. If $\gamma = \gamma'$, the hook lengths on the main diagonal are distinct odd integers summing to n ; their decreasing sequence is a partition $\delta(\gamma)$ of n , the principal-hook partition. The same distinct-odd partitions are precisely the cycle types whose S_n -conjugacy classes split in A_n . Pak, Panova and Vallejo exploited this coincidence in their Main Lemma: if $\chi^\lambda(\delta(\gamma)) \neq 0$, then $g(\lambda, \gamma, \gamma) > 0$ [3]. Their result supplies a powerful nonvanishing certificate for tensor squares.

The A_n splitting mechanism itself is classical and already enters the proof of the Pak–Panova–Vallejo Main Lemma. The point of the present paper is instead to retain the information that is lost when the two split channels are recombined and one passes immediately to positivity. For self-conjugate γ , the restriction $\chi^\gamma \downarrow_{A_n}$ splits as two irreducibles. Keeping these constituents separate reveals two genuine fusion multiplicities whose sum is a Kronecker coefficient and whose difference is the principal-hook character value. Thus the exceptional value $\chi^\lambda(\delta(\gamma))$ is not merely a test for nonvanishing: it is an exact imbalance between two A_n tensor-product channels.

More precisely, enumerate the self-conjugate partitions of n as γ_i , let $\delta_i = \delta(\gamma_i)$, and write

$$\text{Res}_{A_n}^{S_n} \chi^{\gamma_i} = \alpha_i + \beta_i, \quad \Delta_i = \alpha_i - \beta_i.$$

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For $\lambda \neq \lambda'$, the restricted character $\psi_\lambda = \text{Res}_{A_n}^{S_n} \chi^\lambda$ is irreducible and invariant under the outer involution induced by an odd permutation. We first prove

$$\psi_\lambda \Delta_i = \chi^\lambda(\delta_i) \Delta_i.$$

If

$$a_{ij} = \langle \psi_\lambda \alpha_i, \alpha_j \rangle = \langle \psi_\lambda \beta_i, \beta_j \rangle, \quad b_{ij} = \langle \psi_\lambda \alpha_i, \beta_j \rangle = \langle \psi_\lambda \beta_i, \alpha_j \rangle,$$

then the eigenvector identity gives $a_{ij} - b_{ij} = \chi^\lambda(\delta_i) \delta_{ij}$, while index-two reciprocity gives $a_{ij} + b_{ij} = g(\lambda, \gamma_i, \gamma_j)$. Solving these equations yields the formulas stated in the abstract.

The diagonal consequence

$$g(\lambda, \gamma_i, \gamma_i) \geq |\chi^\lambda(\delta_i)|$$

is an exact multiplicity-level refinement of the earlier nonvanishing implication for non-self-conjugate λ : it preserves magnitude, sign through channel imbalance, and parity. The relevant distinction is therefore not a new use of A_n -splitting, but the simultaneous bookkeeping of both split fusion channels. Related refinements of Kronecker squares into symmetric and antisymmetric tensor-square multiplicities have been studied by Bessenrodt and Bowman [4]; those multiplicities are different from the A_n restriction channels considered here. On the class-algebra side, Tekin and Katriel [5] showed that structure constants involving split A_n classes are controlled by character values attached to self-conjugate diagrams and distinct-odd cycle types. Their setting concerns products of class sums rather than the tensor-product multiplicities studied below. The matrix formulation is equally structural. With $A = (a_{ij})$, $B = (b_{ij})$, $G_\lambda = (g(\lambda, \gamma_i, \gamma_j))$ and $X_\lambda = \text{diag}(\chi^\lambda(\delta_i))$, one has $A + B = G_\lambda$ and $A - B = X_\lambda$. Hence the split-fusion block $\begin{pmatrix} A & B \\ B & A \end{pmatrix}$ decomposes into invariant and anti-invariant sectors $G_\lambda \oplus X_\lambda$.

The paper is organized in five sections. Section 2 first isolates the abstract index-two split-fusion lemma and then specializes the required support and reciprocity identities to S_n/A_n . Section 3 proves the exact split-fusion multiplicities, characterizes equality in the principal-hook bound, and gives complete $n = 5$ and genuinely two-dimensional $n = 8$ computations. Section 4 gives the spectral block decomposition and, rather than treating arbitrary nonnegative blocks as faithful models, separates the necessary arithmetic and representation-ring constraints satisfied by realizable fusion matrices. Section 5 analyzes the self-conjugate multiplying-character case and explains precisely why the two-channel scalar formula is replaced by a four-channel system.

2. PRELIMINARIES AND INDEX-TWO RECIPROCITY

The mechanism behind the main theorem is not peculiar to symmetric groups. We begin with an abstract index-two statement that separates the genuinely general Clifford-theoretic input from the special principal-hook geometry of S_n/A_n .

Theorem 2.1 (Abstract index-two split-fusion principle). *Let $H \triangleleft G$ have index two and let $t \in G \setminus H$. Suppose $\Gamma_i, \Gamma_j \in \text{Irr}(G)$ restrict as*

$$\text{Res}_H^G \Gamma_k = \alpha_k + \beta_k, \quad \alpha_k^t = \beta_k \quad (k = i, j),$$

with $\alpha_k, \beta_k \in \text{Irr}(H)$ distinct, and put $\Delta_k = \alpha_k - \beta_k$. Let $\Lambda \in \text{Irr}(G)$ have irreducible restriction $\psi = \text{Res}_H^G \Lambda$. Assume that

$$\psi \Delta_i = c_i \Delta_i$$

for a scalar c_i . Define

$$a_{ij} = \langle \psi \alpha_i, \alpha_j \rangle_H, \quad b_{ij} = \langle \psi \alpha_i, \beta_j \rangle_H.$$

Then

$$a_{ij} + b_{ij} = \langle \Lambda \Gamma_i, \Gamma_j \rangle_G, \quad a_{ij} - b_{ij} = c_i \delta_{ij},$$

and hence

$$a_{ij} = \frac{\langle \Lambda \Gamma_i, \Gamma_j \rangle_G + c_i \delta_{ij}}{2}, \quad b_{ij} = \frac{\langle \Lambda \Gamma_i, \Gamma_j \rangle_G - c_i \delta_{ij}}{2}.$$

In particular the ambient tensor multiplicity is at least $|c_i|$ on the diagonal and is even off the diagonal.

Proof. Because ψ is the restriction of a G -character, it is invariant under conjugation by t . Thus

$$\langle \psi \alpha_i, \alpha_j \rangle_H = \langle \psi \beta_i, \beta_j \rangle_H, \quad \langle \psi \alpha_i, \beta_j \rangle_H = \langle \psi \beta_i, \alpha_j \rangle_H.$$

Expanding $\langle \psi \Delta_i, \Delta_j \rangle_H$ and using orthogonality of the irreducible constituents gives

$$2(a_{ij} - b_{ij}) = \langle c_i \Delta_i, \Delta_j \rangle_H = 2c_i \delta_{ij}.$$

Let ε be the nontrivial character of G/H . Since $\Gamma_j \otimes \varepsilon \cong \Gamma_j$ whenever its restriction splits, index-two reciprocity gives

$$\langle \psi(\alpha_i + \beta_i), \alpha_j + \beta_j \rangle_H = 2\langle \Lambda \Gamma_i, \Gamma_j \rangle_G.$$

The left side is $2(a_{ij} + b_{ij})$. Solving the sum and difference equations proves the result. $\square$

The eigenvector hypothesis in Theorem 2.1 is substantive: it need not hold for an arbitrary index-two extension. What is special for S_n/A_n is that each relevant difference character is supported on a single pair of split classes, while every restricted S_n -character is constant on that pair. This supplies the scalar c_i canonically as a principal-hook character value.

We now use standard facts on the representation theory of S_n and A_n ; see [1, 6]. For $\rho \vdash n$, tensoring with the sign character conjugates the partition:

$$\chi^{\rho'} = \chi^\rho \otimes \text{sgn}.$$

Since sgn is trivial on A_n , conjugate S_n -characters have identical restrictions to A_n . Clifford theory gives the familiar dichotomy: if $\rho \neq \rho'$, then $\text{Res}_{A_n}^{S_n} \chi^\rho$ is irreducible; if $\rho = \rho'$, then it is the sum of two distinct irreducibles interchanged by conjugation with any odd permutation.

A class of S_n splits in A_n exactly when its cycle type has distinct odd parts. For self-conjugate γ_i , its principal-hook partition δ_i is such a cycle type. Denote the two A_n classes below it by $C_{\delta_i}^+$ and $C_{\delta_i}^-$. The difference character $\Delta_i = \alpha_i - \beta_i$ is anti-invariant under the outer involution.

Lemma 2.2 (Support and orthogonality of split differences). *The character Δ_i is supported on $C_{\delta_i}^+ \cup C_{\delta_i}^-$ and takes opposite values on the two halves. Moreover*

$$\langle \Delta_i, \Delta_j \rangle_{A_n} = 2\delta_{ij}.$$

Proof. On every non-split A_n class, conjugation by an odd permutation fixes the class and interchanges α_i and β_i , so their values coincide. On the two halves of the split class the same outer involution exchanges the halves, forcing opposite difference values. Orthogonality follows because α_i, β_i are distinct irreducibles and the split pairs arising from different self-conjugate partitions are disjoint. $\square$

Proposition 2.3 (Difference-character eigenvector identity). *If $\lambda \neq \lambda'$, then for every i ,*

$$\psi_\lambda \Delta_i = \chi^\lambda(\delta_i) \Delta_i, \quad \langle \psi_\lambda \Delta_i, \Delta_j \rangle_{A_n} = 2\chi^\lambda(\delta_i) \delta_{ij}.$$

Proof. The two classes in the support of Δ_i fuse to the single S_n class of cycle type δ_i . Hence ψ_λ has the constant value $\chi^\lambda(\delta_i)$ there, while both sides vanish off the support. The inner-product formula follows from Lemma 2.2. $\square$

For completeness we record the index-two identity that supplies the complementary sum relation.

Lemma 2.4 (Index-two reciprocity). *Let $H \triangleleft G$ have index two and let ε be the nontrivial character of G/H , inflated to G . For class functions f, g on G ,*

$$\langle \text{Res}_H^G f, \text{Res}_H^G g \rangle_H = \langle f, g \rangle_G + \langle f, g\varepsilon \rangle_G.$$

Proof. Since $\text{Ind}_H^G \mathbf{1}_H = \mathbf{1}_G + \varepsilon$, Frobenius reciprocity gives

$$\langle \text{Res } f, \text{Res } g \rangle_H = \langle f, \text{Ind Res } g \rangle_G = \langle f, g(\mathbf{1}_G + \varepsilon) \rangle_G.$$

$\square$

Apply Lemma 2.4 with $G = S_n$, $H = A_n$, $\varepsilon = \text{sgn}$, $f = \chi^\lambda \chi^{\gamma_i}$ and $g = \chi^{\gamma_j}$. Since $\gamma_j = \gamma_j'$, one has $\chi^{\gamma_j} \text{sgn} = \chi^{\gamma_j'}$, and therefore

$$\langle \psi_\lambda(\alpha_i + \beta_i), \alpha_j + \beta_j \rangle_{A_n} = 2g(\lambda, \gamma_i, \gamma_j).$$

Outer invariance pairs the four summands into two equal pairs. This is the precise index-two mechanism used below.

3. MAIN SPLIT-FUSION MULTIPLICITIES

We now retain the two split constituents rather than recombining them. Define a_{ij} and b_{ij} as in the introduction. They are nonnegative integers, and outer invariance gives the displayed equalities in their definitions.

Theorem 3.1 (Exact split-fusion formula). *Let $\lambda \neq \lambda'$ and let γ_i, γ_j be self-conjugate partitions of n . Then*

$$\boxed{a_{ij} = \frac{g(\lambda, \gamma_i, \gamma_j) + \chi^\lambda(\delta_i)\delta_{ij}}{2}}, \quad \boxed{b_{ij} = \frac{g(\lambda, \gamma_i, \gamma_j) - \chi^\lambda(\delta_i)\delta_{ij}}{2}}.$$

Proof. Expanding $\langle \psi_\lambda \Delta_i, \Delta_j \rangle$ and using outer invariance gives

$$2(a_{ij} - b_{ij}) = 2\chi^\lambda(\delta_i)\delta_{ij}.$$

On the other hand, the final identity of Section 2 expands to

$$2(a_{ij} + b_{ij}) = 2g(\lambda, \gamma_i, \gamma_j).$$

Adding and subtracting proves the formulas. □

Corollary 3.2 (Magnitude, parity, and off-diagonal evenness). *For every self-conjugate γ_i ,*

$$g(\lambda, \gamma_i, \gamma_i) \geq |\chi^\lambda(\delta_i)|, \quad g(\lambda, \gamma_i, \gamma_i) \equiv \chi^\lambda(\delta_i) \pmod{2}.$$

If $i \neq j$, then $g(\lambda, \gamma_i, \gamma_j)$ is even and

$$a_{ij} = b_{ij} = \frac{1}{2}g(\lambda, \gamma_i, \gamma_j).$$

Proof. On the diagonal, Theorem 3.1 expresses $(g \pm \chi)/2$ as nonnegative integers. Off the diagonal the character-value correction vanishes. □

The exact formulas also identify the amount by which the principal-hook bound fails to be sharp. This gives a useful equality criterion that is invisible in a mere nonvanishing statement.

Proposition 3.3 (Fusion defect and equality criterion). *For a self-conjugate γ_i , put*

$$x_i = \chi^\lambda(\delta_i), \quad g_i = g(\lambda, \gamma_i, \gamma_i), \quad d_i = \frac{g_i - |x_i|}{2}.$$

Then d_i is a nonnegative integer and

$$d_i = \min\{a_{ii}, b_{ii}\}.$$

Consequently

$$g(\lambda, \gamma_i, \gamma_i) = |\chi^\lambda(\delta_i)|$$

holds if and only if one of the two diagonal split-fusion channels vanishes. More precisely, if $x_i > 0$ equality is equivalent to $b_{ii} = 0$; if $x_i < 0$ it is equivalent to $a_{ii} = 0$; and if $x_i = 0$ equality is equivalent to $a_{ii} = b_{ii} = 0$.

Proof. Theorem 3.1 gives $a_{ii} = (g_i + x_i)/2$ and $b_{ii} = (g_i - x_i)/2$. If $x_i \geq 0$, then $b_{ii} = (g_i - |x_i|)/2$ and $a_{ii} \geq b_{ii}$; if $x_i \leq 0$, the roles are reversed. The asserted equality conditions follow immediately. □

Thus the lower bound has a canonical even defect,

$$g(\lambda, \gamma_i, \gamma_i) - |\chi^\lambda(\delta_i)| = 2d_i,$$

and d_i measures the multiplicity shared by the two competing diagonal channels. Equality is therefore exactly the one-sided-fusion regime.

Explicit examples. Take $n = 5$, $\gamma = (3, 1, 1)$ and $\lambda = (4, 1)$. The partition γ is self-conjugate and its principal-hook partition is $\delta = (5)$. The character $\chi^{(4,1)}$ is the standard character, so at a 5-cycle

$$\chi^{(4,1)}(5) = \# \text{Fix}(5\text{-cycle}) - 1 = -1.$$

For the standard representation one has the classical corner formula

$$g((n-1, 1), \mu, \mu) = c(\mu) - 1,$$

where $c(\mu)$ is the number of removable corners of μ . The diagram $(3, 1, 1)$ has two removable corners, hence

$$g((4, 1), (3, 1, 1), (3, 1, 1)) = 1.$$

Theorem 3.1 therefore gives

$$a = \frac{1 + (-1)}{2} = 0, \quad b = \frac{1 - (-1)}{2} = 1.$$

Thus, in this one-dimensional self-conjugate index set,

$$A = (0), \quad B = (1), \quad G = (1), \quad X = (-1).$$

The full split-fusion matrix on the ordered basis (α, β) is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Passing to the invariant and anti-invariant vectors $\alpha + \beta$ and $\alpha - \beta$ gives eigenvalues 1 and -1 , respectively. Hence the decomposition is exactly

$$G \oplus X = (1) \oplus (-1),$$

and the negative principal-hook value records a one-unit surplus in the crossed channel b .

The first genuinely matrix-valued case already occurs at $n = 8$. There are two self-conjugate partitions,

$$\gamma_1 = (4, 2, 1, 1), \quad \gamma_2 = (3, 3, 2),$$

with principal-hook partitions

$$\delta_1 = (7, 1), \quad \delta_2 = (5, 3).$$

Take $\lambda = (6, 2)$. A direct S_8 character-table computation (equivalently, the Murnaghan–Nakayama rule together with character orthogonality) gives

$$\chi^{(6,2)}(7, 1) = -1, \quad \chi^{(6,2)}(5, 3) = 0,$$

and

$$G_\lambda = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}.$$

Hence

$$X_\lambda = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A = \frac{G_\lambda + X_\lambda}{2} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \frac{G_\lambda - X_\lambda}{2} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}.$$

This example displays both phenomena simultaneously: the off-diagonal Kronecker coefficient is the even integer $g((6, 2), \gamma_1, \gamma_2) = 2$, with $a_{12} = b_{12} = 1$, while the two diagonal principal-hook bounds are strict. Indeed their fusion defects are $d_1 = 2$ and $d_2 = 1$. The full split-fusion operator is

$$\mathcal{M}_{(6,2)} = \begin{pmatrix} 2 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 \\ 3 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

and the invariant/anti-invariant change of basis gives

$$\mathcal{M}_{(6,2)} \sim G_\lambda \oplus X_\lambda.$$

Since G_λ has eigenvalues 6 and 1, the spectrum of the full operator is $\{6, 1, -1, 0\}$. Thus the $n = 8$ computation is not merely a dimensional illustration: it exhibits a nonzero off-diagonal fusion channel, strict diagonal defects, and the spectral separation of the two sectors in a single example.

4. SPECTRAL INTERPRETATION AND MATRIX CONSTRAINTS

Let the indices range over all self-conjugate partitions of n , and set

$$A = (a_{ij}), \quad B = (b_{ij}), \quad G_\lambda = (g(\lambda, \gamma_i, \gamma_j)), \quad X_\lambda = \text{diag}(\chi^\lambda(\delta_i)).$$

Theorem 3.1 is equivalent to

$$A + B = G_\lambda, \quad A - B = X_\lambda, \quad G_\lambda = 2B + X_\lambda.$$

The multiplication operator by ψ_λ on the span of the split constituents has matrix

$$\mathcal{M}_\lambda = \begin{pmatrix} A & B \\ B & A \end{pmatrix}.$$

The orthogonal change of coordinates

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ I & -I \end{pmatrix}$$

yields

$$Q^\top \mathcal{M}_\lambda Q = \begin{pmatrix} G_\lambda & 0 \\ 0 & X_\lambda \end{pmatrix}.$$

Thus the invariant sector is the self-conjugate Kronecker block, while the anti-invariant sector is diagonal and has spectrum precisely $\{\chi^\lambda(\delta_i)\}_i$. This is an exact representation-theoretic decomposition rather than a numerical analogy.

The channel formulas also impose a global congruence on the entire Kronecker block, not only on its diagonal entries.

Theorem 4.1 (Parity profile of the self-conjugate Kronecker block). *With the notation above,*

$$G_\lambda \equiv X_\lambda \pmod{2}$$

entrywise. Consequently, over $\mathbf{F}_2$,

$$\text{rank}_{\mathbf{F}_2}(G_\lambda) = \#\{i : \chi^\lambda(\delta_i) \text{ is odd}\},$$

and the characteristic polynomial satisfies

$$\det(tI - G_\lambda) \equiv \prod_i (t - \chi^\lambda(\delta_i)) \pmod{2}.$$

More generally, for every index subset I ,

$$\det G_\lambda[I] \equiv \prod_{i \in I} \chi^\lambda(\delta_i) \pmod{2},$$

where $G_\lambda[I]$ denotes the corresponding principal submatrix. In particular, $\det G_\lambda$ is odd if and only if every principal-hook value $\chi^\lambda(\delta_i)$ is odd.

Proof. The identities $G_\lambda - X_\lambda = 2B$ and $G_\lambda + X_\lambda = 2A$ imply $G_\lambda \equiv X_\lambda \pmod{2}$. Reduction modulo 2 therefore identifies G_λ with a diagonal matrix whose diagonal entries are the reductions of the principal-hook values. The rank and characteristic-polynomial statements follow immediately, and the same argument applied to each principal submatrix gives the minor congruence. $\square$

Thus the parity pattern of all principal-hook values determines the complete reduction of the self-conjugate Kronecker block modulo 2. This packages the diagonal parity congruence and off-diagonal evenness of Corollary 3.2 into a matrix-level invariant.

The preceding identities also clarify what an actual split-fusion block must satisfy. At the weakest level, A and B are symmetric entrywise-nonnegative integral matrices. The exact formulas impose the stronger arithmetic conditions

$$G_\lambda \pm X_\lambda \in 2M_q(\mathbb{Z}_{\geq 0}),$$

so $G_\lambda \equiv X_\lambda \pmod{2}$ and $(G_\lambda)_{ii} \geq |(X_\lambda)_{ii}|$. More importantly, A and B are not arbitrary matrices satisfying these conditions: their entries are tensor-product structure constants in the representation ring $R(A_n)$ for multiplication by the genuine character ψ_λ . Consequently the full multiplication matrix of ψ_λ on $R(A_n)$ is a nonnegative integral matrix satisfying all associativity relations of the based ring, and the

displayed block is a principal compression of that multiplication operator. Thus there is a strict hierarchy of constraints

$$\text{nonnegative block grammar} \supseteq \text{integral parity-compatible blocks} \supseteq \text{blocks realizable in } R(A_n).$$

The last inclusion encodes compatibility with every other tensor-product multiplication operator, not merely with $A - B = X_\lambda$. A complete intrinsic characterization of the final class appears to be a separate realizability problem.

It is tempting to seek stronger inertia information about G_λ from the first, purely matrix-theoretic level alone. That abstraction is insufficient. Consider

$$B = \begin{pmatrix} 4 & 4 \\ 4 & 1 \end{pmatrix}, \quad X = \text{diag}(-1, -1), \quad A = B + X = \begin{pmatrix} 3 & 4 \\ 4 & 0 \end{pmatrix}.$$

Both A and B are entrywise nonnegative and $A - B = X$ is diagonal, exactly matching the formal matrix grammar. Nevertheless

$$G = A + B = 2B + X = \begin{pmatrix} 7 & 8 \\ 8 & 1 \end{pmatrix}$$

has eigenvalues $4 \pm \sqrt{73}$, approximately 12.544 and -4.544 . Two negative diagonal entries of X therefore coexist with only one nonnegative eigenvalue of G . Any global inertia theorem must use additional fusion-ring or representation-theoretic structure, not merely entrywise nonnegativity.

The same distinction clarifies the relation with [3]. Their proof already uses the classical splitting of self-conjugate characters on restriction to A_n ; thus the use of the index-two subgroup is not asserted here as a new mechanism. The refinement is that the two resulting fusion channels are retained simultaneously. Their sum is the Kronecker multiplicity and their difference is the full integer principal-hook value. Consequently the absolute-value bound, parity congruence, and off-diagonal evenness follow from two-channel integrality. In a targeted literature search, we did not locate these exact formulas in this form; nevertheless, because equivalent identities may occur under different notation, the claim made here is the mathematical refinement itself rather than a categorical priority claim. The tensor-square splitting of Bessenrodt–Bowman [4] and the split-class algebra formulas of Tekin–Katriel [5] are nearby but concern different decompositions. Theorem 2.1 gives the precise index-two extension of the mechanism; the identification of its eigenvalues with principal-hook character values is special to S_n/A_n .

5. THE SELF-CONJUGATE MULTIPLIER AND FURTHER DIRECTIONS

The hypothesis $\lambda \neq \lambda'$ is exactly what makes the preceding theory two-channel. We now make precise what changes when $\lambda = \lambda'$. Write

$$\text{Res}_{A_n}^{S_n} \chi^\lambda = \rho^+ + \rho^-, \quad (\rho^+)^t = \rho^-$$

for an odd permutation t , and retain $\text{Res} \chi^{\gamma_i} = \alpha_i + \beta_i$. For a fixed sign $\varepsilon \in \{+, -\}$ multiplication by ρ^ε has four natural channels

$$u_{ij}^\varepsilon = \langle \rho^\varepsilon \alpha_i, \alpha_j \rangle, \quad v_{ij}^\varepsilon = \langle \rho^\varepsilon \alpha_i, \beta_j \rangle, \quad w_{ij}^\varepsilon = \langle \rho^\varepsilon \beta_i, \alpha_j \rangle, \quad z_{ij}^\varepsilon = \langle \rho^\varepsilon \beta_i, \beta_j \rangle.$$

Conjugation by t does not preserve a fixed multiplier: it exchanges ρ^+ and ρ^- . Hence

$$u_{ij}^+ = z_{ij}^-, \quad v_{ij}^+ = w_{ij}^-, \quad w_{ij}^+ = v_{ij}^-, \quad z_{ij}^+ = u_{ij}^-.$$

The ordinary Kronecker coefficient controls only the total mass of these four channels.

Proposition 5.1 (Four-channel mass identity). *If $\lambda, \gamma_i, \gamma_j$ are self-conjugate, then*

$$u_{ij}^+ + v_{ij}^+ + w_{ij}^+ + z_{ij}^+ = g(\lambda, \gamma_i, \gamma_j),$$

and the same identity holds with $+$ replaced by $-$.

Proof. Index-two reciprocity applied to $f = \chi^\lambda \chi^{\gamma_i}$ and $g = \chi^{\gamma_j}$ gives

$$\langle (\rho^+ + \rho^-)(\alpha_i + \beta_i), \alpha_j + \beta_j \rangle_{A_n} = 2g(\lambda, \gamma_i, \gamma_j).$$

Conjugation by an odd permutation exchanges the complete $+$ and $-$ contributions, so they are equal. Each must therefore equal $g(\lambda, \gamma_i, \gamma_j)$, which is precisely the asserted four-term sum. $\square$

This proposition explains why Theorem 3.1 does not extend by simply replacing ψ_λ with one of $\rho^\pm$. In the non-self-conjugate case the multiplier is outer-invariant, forcing the four matrix entries into two paired channels and allowing the difference-character eigenvalue to determine their difference. For a self-conjugate multiplier, the outer involution exchanges the two multiplication operators themselves. A second scalar identity would require the actual split values of $\rho^\pm$ on the relevant classes. Classical formulas show that these values contain square-root terms on the split class corresponding to λ ; see, for example, the alternating-group character formulas in [1]. Thus the natural extension is genuinely a coupled four-channel problem rather than a missing special case of the two-channel theorem.

The results above isolate a broader principle. The abstract Theorem 2.1 shows that an index-two extension yields exact half-sum/half-difference multiplicities whenever the anti-invariant difference sector is diagonalized by the restricted multiplier. The pair S_n/A_n is especially effective because principal hooks identify the relevant split classes and make the eigenvalues ordinary S_n character values. Within this specialization, the fusion-defect formula characterizes sharpness of the principal-hook bound, while $G_\lambda \equiv X_\lambda \pmod{2}$ determines the complete self-conjugate Kronecker block modulo 2.

Two problems remain natural. First, one may ask for an intrinsic characterization of those parity-compatible pairs (G, X) that are realizable by A_n fusion, incorporating the associativity constraints of $R(A_n)$. Second, in the self-conjugate multiplier case, the four-channel mass identity above should be combined with the explicit split-character values to determine which additional linear combinations of u, v, w, z are forced by the principal-hook geometry. These questions are representation-theoretic rather than purely matrix-theoretic and delineate the point at which the elementary index-two argument gives way to finer structure.

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