

STRUCTURAL PROGRESS ON OPAC-038: SIGN FILTERS, BINARY COMPRESSION, SPLIT FUSION, AND COMPLEMENTARY MINORS

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ABSTRACT. For an irreducible character χ^λ of the symmetric group S_n , let

$$N_n^-(\lambda) = \#\{\mu \vdash n : \chi^\lambda(\mu) < 0\},$$

with conjugacy classes counted uniformly. OPAC-038 asks whether, for all sufficiently large n , this statistic is maximized by the sign representation. The problem remains open.

We give a standalone synthesis of exact structural advances toward the problem. The sign row has $(p(n) - q(n))/2$ negative entries, where $q(n)$ is the number of self-conjugate partitions, and extremality is equivalent to a parity-balance inequality. A known vanishing theorem for self-conjugate characters implies that self-conjugate rows satisfy the desired bound for all sufficiently large n . An explicit rising-factorial character filter converts the sign problem exactly into a rank/support lower bound for a genuine positive-semidefinite fusion matrix. We identify a canonical three-sector decomposition of the representation ring, of dimensions r, r, q , corresponding to odd classes, even non-distinct-odd classes, and distinct-odd classes.

For balanced hooks we derive an exact formula across canonical Glaisher edges, prove injective positivity above the one-third threshold, classify all strict negative sources in the next range, construct parity-complete all-odd absorption targets, and exhibit an infinite family for which every canonical two-part odd refinement fails. On the distinct-odd sector, restriction to A_n yields exact split-fusion multiplicities and the bound $g(\lambda, \gamma, \gamma) \geq |\chi^\lambda(\delta)|$. Conjugation modules and Schur-positive class sums give genuinely unweighted moment domination and constituent-level nonnegative localizers.

Finally, the positive filter admits an adaptive oriented-transversal certificate. After passing to conjugation-adapted columns, a Laplace expansion turns this certificate into simultaneous nonvanishing of complementary odd and even character-table minors, equivalently a representable matroid-capacity problem. A fixed orientation is impossible already in S_4 . This complementary-minor problem is the sharpest reduction obtained here; it is sufficient for OPAC-038 but is not proved. We state all open assertions explicitly and record several no-go results that rule out scalar moments, generic matrix positivity, one-reservoir matching, and antisymmetric-only Kronecker certificates as complete arguments.

1. INTRODUCTION

The irreducible characters and conjugacy classes of S_n are indexed by partitions of n . We write $\chi^\lambda(\mu)$ for the value of the row indexed by $\lambda \vdash n$ on the class of cycle type $\mu \vdash n$. The statistic

$$N_n^-(\lambda) = \#\{\mu \vdash n : \chi^\lambda(\mu) < 0\}$$

is deliberately unweighted: every conjugacy class contributes one unit, irrespective of its size or centralizer.

Let $p(n)$ be the partition number and let $q(n)$ be the number of self-conjugate partitions of n , equivalently the number of partitions of n into distinct odd parts. Put

$$r_n = \frac{p(n) - q(n)}{2}, \quad h_n = \frac{p(n) + q(n)}{2}.$$

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The sign row has exactly r_n negative entries. OPAC-038 asks whether

$$(1) \quad N_n^-(\lambda) \leq r_n$$

for every $\lambda \vdash n$ once n is sufficiently large.

The central obstacle is the mismatch between the desired unweighted sign count and the weighted information naturally produced by orthogonality, central characters, and class algebra. The purpose of this paper is to assemble the strongest exact consequences presently available into one coherent architecture while keeping theorem, reformulation, finite evidence, and open target logically separate.

The main contributions are as follows. First, parity reduces (1) to an exact comparison between negative even classes and nonpositive odd classes. Second, a known vanishing theorem of Kawsathon–Rodtes [8] removes self-conjugate rows asymptotically. Third, a genuine-character polynomial filter removes magnitudes altogether and turns the target into a rank/support inequality. Fourth, balanced hooks admit an exact binary-compression analysis that supplies explicit payments and explicit obstructions. Fifth, the distinct-odd exceptional sector is identified with the outer anti-invariant sector of A_n , where exact split-fusion multiplicities strengthen principal-hook nonvanishing mechanisms related to those in Pak–Panova–Vallejo [10]. Sixth, Sundaram’s Schur-positive class sums [13] and the conjugation action yield exact unweighted moment and constituent-level inequalities. Finally, the filtered-rank formulation is reduced to an adaptive oriented-transversal problem and then to complementary nonzero minors, giving a concrete matroid-capacity frontier.

No result below proves OPAC-038. In particular, the oriented-transversal and complementary-minor statements are sufficient targets, not established theorems.

2. PARITY AND THE EXACT FINITE-DEGREE TARGET

For $\mu = (1^{m_1} 2^{m_2} \dots) \vdash n$, put

$$z_\mu = \prod_{i \geq 1} i^{m_i} m_i!, \quad \varepsilon(\mu) = (-1)^{n - \ell(\mu)}.$$

Let E_n and O_n be the even and odd conjugacy classes. The sign twist is

$$(2) \quad \chi^{\lambda'}(\mu) = \varepsilon(\mu) \chi^\lambda(\mu).$$

Proposition 2.1 (Sign-row count). *For every n ,*

$$\sum_{\mu \vdash n} \varepsilon(\mu) = q(n), \quad N_n^-((1^n)) = r_n.$$

Proof. The signed class generating function is

$$\prod_{j \text{ odd}} (1 - x^j)^{-1} \prod_{j \text{ even}} (1 + x^j)^{-1} = \prod_{r \geq 1} (1 + x^{2r-1}),$$

the generating function for distinct-odd partitions. If P_n, M_n are the positive and negative sign-row counts, then $P_n + M_n = p(n)$ and $P_n - M_n = q(n)$. $\square$

For a fixed λ , write $E^-(\lambda)$ for the number of negative values on even classes and $O^-(\lambda), O^0(\lambda), O^+(\lambda)$ for the corresponding odd-class counts.

Proposition 2.2 (Parity balance). *For every $\lambda \vdash n$,*

$$N_n^-(\lambda') - r_n = E^-(\lambda) - O^-(\lambda) - O^0(\lambda).$$

Consequently, sign-row maximality in degree n is equivalent to

$$(3) \quad E^-(\lambda) \leq O^-(\lambda) + O^0(\lambda) \quad (\lambda \vdash n).$$

Proof. By (2), $N_n^-(\lambda') = E^-(\lambda) + O^+(\lambda)$. Since every odd class is negative in the sign row, $r_n = O^-(\lambda) + O^0(\lambda) + O^+(\lambda)$. Subtract. $\square$

Let DO_n denote the distinct-odd partitions and $B_n = E_n \setminus \text{DO}_n$. Then $|B_n| = |O_n| = r_n$. If D^- is the negative count on DO_n and B^0, B^+ are the zero and positive counts on B_n , then (3) is equivalently

$$(4) \quad D^- + O^+ \leq B^0 + B^+.$$

This is the first capacity form of the problem: negative exceptional values and positive odd values must be paid for by nonnegative capacity in the balanced even reservoir.

3. VANISHING AND ASYMPTOTIC ELIMINATION OF SELF-CONJUGATE ROWS

We record two support restrictions. The first is standard from Murnaghan–Nakayama.

Lemma 3.1 (Hook-support criterion). *If $\chi^\lambda(\mu) \neq 0$, then every part of μ is a hook length of λ .*

Proof. Apply Murnaghan–Nakayama with a chosen part r of μ first. A nonzero term requires a removable rim hook of length r , equivalently a bead move of size r in a beta-set model, hence a cell of hook length r . $\square$

For self-conjugate diagrams, Kawsathon–Rodtes prove a stronger vanishing phenomenon [8].

Proposition 3.2 (Kawsathon–Rodtes vanishing). *If $\lambda = \lambda' \vdash n$ and a cycle type μ has an even part $h > n/2$, then $\chi^\lambda(\mu) = 0$.*

Theorem 3.3 (Self-conjugate rows satisfy OPAC asymptotically). *For all sufficiently large n , every self-conjugate $\lambda \vdash n$ satisfies*

$$N_n^-(\lambda) \leq r_n.$$

Proof. If $\lambda = \lambda'$, then (2) forces $\chi^\lambda(\mu) = 0$ on every odd class, so $N_n^-(\lambda) = E^-(\lambda)$. Let H_n be the least even integer strictly larger than $n/2$ and put $s_n = n - H_n = n/2 + O(1)$. For each $\nu \vdash s_n$, the class $(H_n) \cup \nu$ is a zero by Theorem 3.2. Because H_n is even, this class is even exactly when ν is odd. Hence the number $Z_E(\lambda)$ of forced even zeros satisfies

$$Z_E(\lambda) \geq \frac{p(s_n) - q(s_n)}{2}.$$

Hardy–Ramanujan asymptotics [6] give $\log p(s_n) = (\pi/\sqrt{3} + o(1))\sqrt{s_n}$, while a direct product bound for distinct-odd partitions gives $\log q(n) \leq (\sqrt{2} + o(1))\sqrt{n}$. Since $\pi/\sqrt{3} > \sqrt{2}$, eventually $Z_E(\lambda) \geq q(n)$. Therefore

$$E^-(\lambda) \leq |E_n| - Z_E(\lambda) \leq h_n - q(n) = r_n.$$

$\square$

Thus any infinite family of counterexamples is eventually non-self-conjugate.

There is also a general support barrier. Let $m(\lambda)$ be the least positive integer that is not a hook length of λ . By Theorem 3.1, a nonzero class can contain no part divisible by $m(\lambda)$. Fristedt’s conditioning method for random partitions [4] yields the following.

Theorem 3.4 (Smallest-missing-hook barrier). *Let c_0 be the unique positive solution of*

$$\prod_{j \geq 1} \left(1 - e^{-\pi c_0 j / \sqrt{6}}\right) = \frac{1}{2} \quad (c_0 = 0.7739525659 \dots).$$

If $\lambda^{(n)} \vdash n$ violates sign-row maximality for infinitely many n , then along that subsequence

$$m(\lambda^{(n)}) \geq (c_0 - o(1))\sqrt{n}.$$

Proof sketch. A violating row has more than $(1/2 - o(1))p(n)$ nonzero entries because $q(n) = o(p(n))$. Its support is contained in the $m(\lambda)$ -class-regular partitions. Fristedt's conditioning theorem gives the limiting density $\prod_{j \geq 1} (1 - e^{-\pi c j / \sqrt{6}})$ when $m/\sqrt{n} \rightarrow c$, and density 0 when $m = o(\sqrt{n})$. The threshold $1/2$ gives c_0 . $\square$

4. EXACT POSITIVE FILTERS AND A SIGN-FREE RANK TARGET

Fix $\lambda \vdash n$ and put $f = \chi^\lambda(1)$. Define

$$Q_f(x) = \prod_{j=1}^f (x + j), \quad P_f(x) = xQ_f(x).$$

Since symmetric-group character values are integers in $[-f, f]$,

$$(5) \quad Q_f(x) = 0 \iff x < 0, \quad Q_f(x) > 0 \iff x \geq 0,$$

on the character spectrum.

Let $K_\lambda = (g(\lambda, \alpha, \beta))_{\alpha, \beta \vdash n}$ be Kronecker multiplication by $[\lambda]$. It is symmetric, entrywise nonnegative and integral, and its class eigenvalues are $\chi^\lambda(\mu)$.

Theorem 4.1 (Genuine-character sign filter). *The representation*

$$\Phi_\lambda = \bigotimes_{j=1}^f (V_\lambda \oplus \mathbf{1}^{\oplus j})$$

has character $Q_f(\chi^\lambda(g)) \geq 0$, with support exactly the classes on which $\chi^\lambda \geq 0$. Consequently

$$F_\lambda := Q_f(K_\lambda)$$

is symmetric, entrywise nonnegative, integral and positive semidefinite, with

$$\text{nullity } F_\lambda = N_n^-(\lambda).$$

Proof. Characters add under direct sum and multiply under tensor product, giving the displayed polynomial. The support follows from (5). Character-table diagonalization of K_λ gives the nullity statement. $\square$

Applying the filter to the conjugate row gives the exact parity-sensitive support

$$S_\lambda = \{\mu \in E_n : \chi^\lambda(\mu) \geq 0\} \sqcup \{\mu \in O_n : \chi^\lambda(\mu) \leq 0\}.$$

Corollary 4.2 (Exact rank reformulation). *For $F_{\lambda'} = Q_f(K_{\lambda'})$,*

$$\text{rank } F_{\lambda'} = |S_\lambda| = h_n - E^-(\lambda) + O^-(\lambda) + O^0(\lambda).$$

Hence (3) is equivalent to

$$(6) \quad \text{rank } F_{\lambda'} \geq h_n.$$

This is an exact reformulation, not a proof. Its advantage is that all character magnitudes have disappeared from the target.

5. POSITIVE CENTRAL FORCING AND LOCALIZED TRANSFER

Let K_μ be the class sum of type μ . Class-sum structure constants are nonnegative integers. In the orthonormal class basis $b_\mu = K_\mu / \sqrt{|C_\mu|}$, multiplication by a real central element with nonnegative class coefficients has an entrywise nonnegative matrix.

For fixed λ , let

$$L_\lambda = \{\alpha \in E_n : \chi^\lambda(\alpha) < 0\}, \quad E_\lambda = \{\beta \in O_n : \chi^\lambda(\beta) \leq 0\}.$$

Theorem 5.1 (Flexible positive forcing). *Let $z = \sum_{\gamma \in O_n} a_\gamma K_\gamma$ with $a_\gamma \geq 0$, and suppose its λ -central eigenvalue is positive. If A_z is the odd-to-even multiplication block, then for every $\alpha \in L_\lambda$,*

$$\sum_{\beta \in E_\lambda} (A_z)_{\alpha\beta} v_\beta^\lambda < 0,$$

where v^λ is the normalized class-coordinate character eigenvector. In particular, $A_z[L_\lambda, E_\lambda]$ has no nonzero left-kernel vector whose coordinates are all nonnegative or all nonpositive.

Proof. Parity reversal gives $A_z v_O^\lambda = d_\lambda(z) v_E^\lambda$. At a negative even coordinate the right side is negative. Contributions from positive odd character coordinates are nonnegative, so the remaining nonpositive odd coordinates must contribute strictly negatively. $\square$

For non-self-conjugate λ , orthogonality implies the odd character vector has both signs, so a single positive odd class sum can always be chosen to activate Theorem 5.1. The remaining obstruction is therefore a mixed-sign dependence, not a one-sign kernel vector.

A complementary global rank statement comes from Jucys–Murphy elements. Write $J_i = \sum_{j < i} (ji)$ and

$$\Omega_t = \prod_{i=1}^n (t + J_i) = \sum_{\sigma \in S_n} t^{\#\text{cyc}(\sigma)} \sigma,$$

using Jucys’ identity [7]. If $P_\rho(t) = \prod_{u \in \rho} (t + c(u))$, the coefficient of t^k is the eigenvalue of the k -cycle-count layer. Frobenius coordinates and a signed-moment argument yield an $O(\sqrt{n})$ bound for the first parity-reversing active layer.

Theorem 5.2 (Square-root localization). *For every non-self-conjugate $\lambda \vdash n$, the first nonzero parity-reversing Jucys layer occurs at an index*

$$\kappa(\lambda) < 3\sqrt{n/2}.$$

Consequently, a suitable positive linear combination of parity-reversing layers with at most $\lceil 3\sqrt{n/2} \rceil - 1$ cycles has kernel exactly the self-conjugate irreducibles and maximal parity-direction rank r_n .

Proof sketch. Write Frobenius coordinates $(a_1, \dots, a_d \mid b_1, \dots, b_d)$ and let $\Delta_c = \#\{i : a_i \geq c\} - \#\{i : b_i \geq c\}$. The quotient of the content polynomials of λ and λ' has logarithm $2 \sum_{j \geq 0} M_{2j+1} t^{2j+1} / (2j+1)$ with $M_r = \sum_c \Delta_c / c^r$. A finite signed-moment rigidity lemma bounds the first nonzero odd moment by the number of sign changes of Δ_c . If h is the Frobenius mismatch count, then $d^2 + h^2 \leq n$ and the first active layer is at most $d + 2h - 1 < 3\sqrt{n/2}$. A sufficiently small positive parameter separates the finitely many non-self-conjugate spectral polynomials simultaneously. $\square$

The theorem gives maximal global parity rank, but coordinate restriction can destroy rank; it does not imply (3).

6. GLAISHER COMPRESSION AND BALANCED HOOKS

Glaisher's operation replaces (m, m) by $(2m)$ and conversely. Acting independently along each odd-base binary chain partitions the conjugacy classes into components. Every nontrivial component contains equally many even and odd permutations; the singleton components are exactly the distinct-odd partitions. Thus DO_n is the exceptional sector for parity balancing.

Consider the balanced hook

$$\lambda_k = (k+1, 1^{k-1}) \vdash 2k, \quad X_k(\mu) = \chi^{\lambda_k}(\mu).$$

For hook characters,

$$(7) \quad \sum_{r=0}^{n-1} \chi^{(n-r, 1^r)}(\mu) t^r = \frac{1}{1+t} \prod_i (1 - (-t)^{\mu_i}).$$

Fix a canonical odd-scale edge

$$A = \tau \cup (m, m), \quad B = \tau \cup (2m),$$

with m odd. Put $q = k - m$, $d = k - 2m$, $|\tau| = 2q$, and let $H_j(\tau)$ denote the degree- j hook coefficient for τ .

Theorem 6.1 (Exact hook values across a Glaisher edge). *For the edge above,*

$$(8) \quad X_k(A) = \varepsilon(\tau) H_d(\tau) - 2(-1)^m H_{q-1}(\tau) + H_{d-1}(\tau),$$

$$(9) \quad X_k(B) = \varepsilon(\tau) H_d(\tau) - H_{d-1}(\tau).$$

Proof. In (7) the edge factors are $(1 - (-t)^m)^2$ and $(1 - t^{2m})$. Extract the coefficient of t^{k-1} and use complementary hook symmetry in degree $2q - 1$. $\square$

If μ has only odd parts, then $X_k(\mu) \geq 0$. Indeed, with $P(t) = \prod_i (1 + t^{\mu_i})$ and $Q = P/(1+t)$, palindromicity gives $2[t^{k-1}]Q = [t^k]P \geq 0$.

Assume the edge is terminal at odd scale m : smaller odd-base chains are Glaisher-trivial and no higher binary part occurs in the m -chain.

Theorem 6.2 (Injective positivity above one third). *If $m > d$ (equivalently $m > k/3$), every terminal edge whose two endpoint values are negative admits an injective same-scale map to an edge whose two endpoint values are strictly positive.*

Proof. Terminality and $|\tau| = 2(m+d) < 4m$ force a negative-negative remainder to have the form $\tau = \sigma \cup (2b)$ with $b > m$ odd and σ all odd. Put $|\sigma| = 2a$ and

$$F(t) = \frac{\prod_{x \in \sigma} (1 + t^x)}{1+t} = \sum_j f_j t^j.$$

With $s = 2a - d = k - 2b$, palindromicity gives the source pair

$$(X_k(A), X_k(B)) = (f_s - f_{s-1}, -(f_{s-1} + f_s)) = (-D, -P)$$

for $D, P > 0$. Split $2b$ into (b, b) . The target pair is $(P + 4C, D)$ with $C = f_{a-1} = f_a \geq 0$. It is strictly positive, and the repeated part b recovers the source. $\square$

The next range is $m \leq d < 2m$, equivalently $k/4 < m \leq k/3$.

Theorem 6.3 (Classification in the first range below one third). *Every strict negative-negative terminal odd source is exactly one of:*

- (a) $\tau = \sigma \cup (2b)$ with $b > m$ odd and σ a nonempty all-odd multiset;
- (b) $\tau = (2b, 2b)$, necessarily $q = 2b$, with endpoint pair $(-2, -2)$.

No other multi-even or higher-binary-level configuration is strictly negative at both endpoints.

Proof. If at least two even parts and an odd part occur, the total odd weight is $< 2(d - m) < d$, forcing $H_{d-1} = H_d = 0$ and hence $X_k(B) = 0$. If there are only even parts, size forces exactly two level-one parts, and direct evaluation gives negativity only in the equal case. A unique even part at binary level at least two similarly leaves too little odd weight and gives $X_k(B) = 0$. $\square$

The double-even family is paid injectively by $(2b, 2b) \mapsto (2b, b, b)$, whose endpoint pair is $(0, 0)$. For the one-even family, write $|\sigma| = 2a$, $a + b = q$, $u = a - b = q - 2b$, $s = 2a - d = m + u$. Then

$$(10) \quad X_k(B) = -(f_{s-1} + f_s), \quad X_k(A) = \Delta f_s + 2\Delta f_u,$$

where $\Delta f_j = f_j - f_{j-1}$. For a canonical odd split $2b = v + w$, one endpoint is automatically nonnegative and the other is

$$(11) \quad B_{v,w} = \Delta f_d + \Delta f_{d-v} + \Delta f_{d-w}.$$

Odd splitting is not universal. More strongly, there is an infinite obstruction.

Theorem 6.4 (Infinite split-dead family). *Let a be odd, $j \geq 1$, $a \geq 4j + 1$, and set*

$$m = 2a + 1, \quad d = 3a + 4j + 2, \quad b = 2a + 2j + 1, \\ \sigma = (1, a, 2a + 4j + 1, 3a + 2).$$

Then $m \leq d < 2m$, the source $\sigma \cup (2b)$ has endpoints $(-1, -1)$, and every canonical two-part odd split of $2b$ has strictly negative B -endpoint.

Proof. The part 1 cancels the denominator in F , leaving a product of three binomials. At the relevant indices $(f_s, f_{s-1}) = (1, 0)$ and $(f_u, f_{u-1}) = (0, 1)$, giving $(-1, -1)$. Since d is odd, the correction indices in (11) are even. A nonnegative split would force a split part equal to 1 or d , but each repeats an existing residual part and violates canonicity. $\square$

Despite split failure, all-odd absorption is abundant.

Theorem 6.5 (Parity-complete absorption). *Let $\tau = \sigma \cup (2b)$ be a strict one-even source in the range $m \leq d < 2m$. For every nonempty submultiset $R \subseteq \sigma$ of weight $w < d$, there is an explicit favorable all-odd target:*

- (i) if $|R|$ is odd, $Z_R = R \cup (2q - w)$ has endpoint pair $(0, 0)$;
- (ii) if $|R|$ is even and d is even, $Z_R = R \cup (d + 1, 2q - w - d - 1)$ has pair $(2H_{q-1}(Z_R), 0)$;
- (iii) if $|R|$ is even and d is odd, $Z_R = R \cup (d, 2q - w - d)$ has pair $(1 + 2H_{q-1}(Z_R), 1)$.

In particular every strict one-even source has at least one favorable absorption target.

Proof. The residual quotient determined by R has degree $w - 1 < d - 1$. The added odd parts lie above the low-degree window, except for the controlled part d in (iii). Substitution into Theorem 6.1 gives the displayed pairs. The middle coefficient is nonnegative by the all-odd observation. Since $\text{wt}(\sigma) < 2d$, some singleton $R = (x)$ has $x < d$. $\square$

The unresolved balanced-hook problem is therefore global collision control: sourcewise favorable targets exist, but no theorem presently proves a matching saturating all strict sources using the union of split and absorption targets.

At even binary scales, edgewise cancellation is replaced by higher-dimensional cells. From (7) one obtains

Lemma 6.6 (One-scale Hadamard descent). *For every partition ρ and positive integer a ,*

$$X_r(\rho \cup (a, a)) - X_r(\rho \cup (2a)) = 2(-1)^{a+1} X_{r-a}(\rho \cup a).$$

Iterating in two directions gives

Theorem 6.7 (Binary-square descent). *For positive integers a, b ,*

$$\begin{aligned} & X_r(\rho \cup (a, a, b, b)) - X_r(\rho \cup (2a, b, b)) \\ & - X_r(\rho \cup (a, a, 2b)) + X_r(\rho \cup (2a, 2b)) = 4(-1)^{a+b} X_{r-a-b}(\rho \cup a \cup b). \end{aligned}$$

This identity is not a positivity theorem, but it shows why even-scale arguments naturally require binary cells rather than isolated edges.

7. THE ALTERNATING-GROUP SPLIT-FUSION BRIDGE

Distinct-odd S_n classes are precisely the classes that split in A_n , and they are indexed by self-conjugate partitions through principal hooks. Let $\gamma_1, \dots, \gamma_q$ be the self-conjugate partitions and δ_i the distinct-odd principal-hook partition associated with γ_i . Write

$$\text{Res}_{A_n}^{S_n} \chi^{\gamma_i} = \alpha_i + \beta_i, \quad \Delta_i = \alpha_i - \beta_i.$$

For non-self-conjugate λ , put $\psi_\lambda = \text{Res}_{A_n}^{S_n} \chi^\lambda$.

Theorem 7.1 (Difference-character eigenvector identity). *For every i ,*

$$\psi_\lambda \Delta_i = \chi^\lambda(\delta_i) \Delta_i.$$

Hence

$$\langle \psi_\lambda \Delta_i, \Delta_j \rangle = 2\chi^\lambda(\delta_i) \delta_{ij}.$$

Proof. Δ_i is supported on the two A_n classes below the split S_n class δ_i and takes opposite values there. The restricted S_n class function ψ_λ is constant on the two halves. $\square$

Define split-fusion multiplicities

$$\begin{aligned} a_{ij} &= \langle \psi_\lambda \alpha_i, \alpha_j \rangle = \langle \psi_\lambda \beta_i, \beta_j \rangle, \\ b_{ij} &= \langle \psi_\lambda \alpha_i, \beta_j \rangle = \langle \psi_\lambda \beta_i, \alpha_j \rangle. \end{aligned}$$

Theorem 7.2 (Exact split-fusion identity). *If $\lambda \neq \lambda'$, then*

$$(12) \quad a_{ij} = \frac{g(\lambda, \gamma_i, \gamma_j) + \chi^\lambda(\delta_i) \delta_{ij}}{2}, \quad b_{ij} = \frac{g(\lambda, \gamma_i, \gamma_j) - \chi^\lambda(\delta_i) \delta_{ij}}{2}.$$

In particular,

$$g(\lambda, \gamma_i, \gamma_i) \geq |\chi^\lambda(\delta_i)|, \quad g(\lambda, \gamma_i, \gamma_i) \equiv \chi^\lambda(\delta_i) \pmod{2},$$

and $g(\lambda, \gamma_i, \gamma_j)$ is even for $i \neq j$.

Proof. Theorem 7.1 gives $a_{ij} - b_{ij} = \chi^\lambda(\delta_i) \delta_{ij}$. Restriction-induction reciprocity across the index-two extension gives $a_{ij} + b_{ij} = g(\lambda, \gamma_i, \gamma_j)$. Solve. Nonnegativity and integrality give the inequalities and parity statements. $\square$

Pak–Panova–Vallejo use the A_n splitting mechanism to obtain principal-hook nonvanishing for Kronecker coefficients [10]. Formula (12) retains both fusion channels and therefore records the exact multiplicity imbalance. If $\chi^\lambda(\delta_i) < 0$, then

$$b_{ii} - a_{ii} = -\chi^\lambda(\delta_i) > 0,$$

so every negative exceptional value carries a canonical crossed-channel surplus witness.

For self-conjugate multipliers $\lambda = \lambda'$, the restriction of χ^λ also splits, and the natural bookkeeping has four channels rather than two. No extension of Theorem 7.2 to that case is asserted.

8. EQUAL-PART COARSENING AND PLETHYSTIC SUPPORT

Equal-part coarsening gives useful Fourier infrastructure. Fix $\lambda \vdash n$, $k, m \geq 1$, and τ with $|\tau| + km = n$. For $\nu \vdash m$, define

$$F(\nu) = \chi^\lambda(\tau \cup k\nu).$$

Then, as a class function on S_m ,

$$(13) \quad F(\nu) = \sum_{\alpha \vdash m} c_\alpha \chi^\alpha(\nu), \quad c_\alpha = \langle s_\lambda, p_\tau s_\alpha[p_k] \rangle \in \mathbb{Z}.$$

This is Fourier inversion in the irreducible character basis; the coefficients can be negative, so the fibre is generally virtual.

The plethystic Murnaghan–Nakayama rule, proved combinatorially by Wildon [14], controls extremal modes. For residual size $r = n - km$, put

$$C_+^\lambda(\tau) = \sum_{\nu \vdash m} \frac{\chi^\lambda(\tau \cup k\nu)}{z_\nu}.$$

Proposition 8.1 (Residual plethystic spectrum). *One has*

$$C_+^\lambda(\tau) = \sum_{\mu \vdash r} \epsilon_\mu^\lambda \chi^\mu(\tau), \quad \epsilon_\mu^\lambda \in \{0, \pm 1\},$$

where ϵ_μ^λ is the plethystic Murnaghan–Nakayama coefficient of s_λ in $s_\mu h_m[p_k]$.

Consequently, if $A(\lambda; k, m) = \#\{\mu : \epsilon_\mu^\lambda \neq 0\}$, Parseval gives

$$\sum_{\tau \vdash r} \frac{|C_+^\lambda(\tau)|^2}{z_\tau} = A(\lambda; k, m).$$

This is exact capacity information, but it retains $1/z_\tau$ weights and therefore does not solve the unweighted sign problem.

9. CROSSING THE WEIGHTED-TO-UNWEIGHTED BARRIER

Let κ_n be the character of the conjugation action of S_n on itself. At a class μ , $\kappa_n(\mu) = z_\mu$. Hence for every $j \geq 0$,

$$R_j(\lambda) = \sum_{\mu \vdash n} \chi^\lambda(\mu)^j = \langle \kappa_n(\chi^\lambda)^j, \mathbf{1} \rangle \geq 0,$$

and after twisting by sign,

$$T_j(\lambda) = \sum_{\mu \vdash n} \text{sgn}(\mu) \chi^\lambda(\mu)^j \geq 0.$$

If E_j, O_j are the unweighted moments over even and odd classes, then

$$(14) \quad E_j \geq |O_j| \quad (j \geq 0).$$

Sundaram’s work on the conjugation action and Schur-positive class sums [13] localizes this to the exact nonexceptional reservoirs. In particular, the sum of power sums over all non-distinct-odd classes is Schur-positive. Thus there is a genuine character Φ_N whose class values are z_μ on $N = B_n \sqcup O_n$ and zero elsewhere. Put

$$B_j = \sum_{\mu \in B_n} \chi^\lambda(\mu)^j.$$

Theorem 9.1 (Balanced unweighted moment domination). *For every $\lambda \vdash n$ and every $j \geq 0$,*

$$B_j \geq |O_j|.$$

Proof. Both Φ_N and $\Phi_N \otimes \text{sgn}$ are genuine. Their multiplicities against $(\chi^\lambda)^j$ are respectively $B_j + O_j$ and $B_j - O_j$, so both are nonnegative. $\square$

Scalar moments are insufficient to control negative support: the abstract multisets $\{-1, 3\}$ and $\{-1, -1\}$ satisfy all corresponding moment inequalities but have the wrong negative-count comparison. Constituent labels must be retained.

Let Φ_B be the Schur-positive selector for B_n and $L_B = K(\Phi_B)$. Then L_B is symmetric, positive semidefinite, entrywise nonnegative and integral, with spectral support exactly B_n and rank r_n . For $M_B(\lambda) = L_B K_\lambda$:

Proposition 9.2 (Constituent-level balanced inequalities). *For every $\alpha, \beta, \lambda \vdash n$ and $j \geq 1$,*

$$H_j(\alpha, \beta; \lambda) := \sum_{\mu \in B_n} z_\mu^{j-1} \chi^\lambda(\mu)^j \chi^\alpha(\mu) \chi^\beta(\mu) = (M_B(\lambda)^j)_{\alpha\beta} \in \mathbb{Z}_{\geq 0}.$$

In particular,

$$\sum_{\mu \in B_n} \chi^\lambda(\mu) \chi^\alpha(\mu) \chi^\beta(\mu) \geq 0.$$

This is stronger than scalar moments, but it still localizes only the balanced even reservoir.

10. CANONICAL THREE-SECTOR GEOMETRY

Let J be the involution $[\rho] \mapsto [\rho']$ on the complexified representation ring, with eigenspaces H_+ and H_- . Their dimensions are h_n and r_n . Let $L_N = K_{\Phi_N}$ be the non-distinct-odd selector and write L_N^+ for its restriction to H_+ .

Theorem 10.1 (Three-sector decomposition). *There is a canonical orthogonal decomposition*

$$R(S_n)_\mathbb{C} = H_- \oplus \text{ran}(L_N^+) \oplus \ker(L_N^+)$$

with dimensions $r_n, r_n, q(n)$, respectively. The three summands have class spectral bases indexed by O_n , B_n , and DO_n . For non-self-conjugate λ , multiplication by $\text{Res}_{A_n}^{S_n} \chi^\lambda$ on the outer anti-invariant sector of $R(A_n)_\mathbb{C}$ is similar to $K_\lambda|_{\ker(L_N^+)}$.

Proof. On H_- , L_N has positive eigenvalue z_μ at every odd class, so its rank is r_n . On H_+ , its positive eigenvalues are exactly the B_n classes and its zero eigenvalues exactly the distinct-odd classes, giving dimensions r_n and $q(n)$. The A_n difference-character basis from Theorem 7.1 has the same distinct-odd eigenvalues as K_λ on the kernel sector. $\square$

Thus the exceptional A_n split-fusion sector is not merely analogous to the missing S_n sector: it is the same distinct-odd spectral sector seen through the index-two restriction. However, orthogonally adjoining the sectors simply reconstructs the original spectrum and gives no new sign-count inequality. A successful proof must couple them through module, support, or minor structure.

11. ORIENTED TRANSVERSALS AND COMPLEMENTARY MINORS

The exact filtered rank target (6) suggests constructing h_n independent columns explicitly. Partition the irreducibles into r_n non-self-conjugate pairs $\{\rho_i, \rho'_i\}$ and $q(n)$ self-conjugate irreducibles $\gamma_1, \dots, \gamma_q$. An orientation σ chooses one representative $c_i \in \{\rho_i, \rho'_i\}$ from each pair and sets

$$C_\sigma = \{c_1, \dots, c_r, \gamma_1, \dots, \gamma_q\}, \quad |C_\sigma| = h_n.$$

Proposition 11.1 (Oriented-transversal sufficiency). *If, for a given λ , there is an orientation σ such that the columns of $F_{\lambda'}$ indexed by C_σ are linearly independent, then $N_n^-(\lambda') \leq r_n$.*

Proof. The selected set contains h_n independent columns, so $\text{rank } F_{\lambda'} \geq h_n$. Apply Theorem 4.2. $\square$

Conjecture 11.2 (Oriented Transversal Theorem (OTT)). *For every relevant λ , there exists an orientation σ for which the columns of $F_{\lambda'}$ indexed by C_σ are linearly independent.*

The statement is deliberately labeled as open.

Let X be the ordinary character table, with rows indexed by conjugacy classes and columns by irreducibles. Character-table diagonalization of $F_{\lambda'}$ and invertible centralizer scalings remove the large filter polynomial.

Theorem 11.3 (Restricted-table form). *For a fixed orientation σ , the selected columns of $F_{\lambda'}$ are independent if and only if*

$$X[S_\lambda, C_\sigma]$$

has full column rank h_n .

Proof. The filtered matrix factors through the class eigenspaces in S_λ with strictly positive diagonal weights. The Gram matrix of selected filtered columns therefore has the same rank as the selected columns of X restricted to S_λ . $\square$

Now use conjugation-adapted columns. For each pair $\{\rho_i, \rho'_i\}$, define

$$u_i = \frac{\chi^{\rho_i} + \chi^{\rho'_i}}{2}, \quad v_i = \frac{\chi^{\rho_i} - \chi^{\rho'_i}}{2}.$$

Then u_i is supported on even classes and v_i on odd classes. Every self-conjugate column vanishes on odd classes. Choosing ρ_i or ρ'_i leaves u_i unchanged and changes only the sign of v_i .

Let $E_f = S_\lambda \cap E_n$ and $O_f = S_\lambda \cap O_n$. Choose $E_0 \subseteq E_f$, $O_0 \subseteq O_f$ with $|E_0| + |O_0| = h_n$, and put $o = |O_0|$. Let U be the even-row matrix of the u_i together with the fixed self-conjugate columns Γ , and let V be the odd-row matrix of the v_i .

Theorem 11.4 (Complementary-minor determinant expansion). *For orientation signs $s = (s_1, \dots, s_r) \in \{\pm 1\}^r$, the determinant of the corresponding $h_n \times h_n$ restricted matrix is*

$$(15) \quad \det M(s) = \sum_{\substack{I \subseteq [r] \\ |I|=o}} \epsilon_I \det V[O_0, I] \det U[E_0, I^c \cup \Gamma] \prod_{i \in I} s_i,$$

where $\epsilon_I \in \{\pm 1\}$ depends only on column order.

Proof. The odd rows see only the signed columns $s_i v_i$ and no self-conjugate columns. Laplace expansion along the odd rows chooses a set I of o pair columns for the odd block; the complementary even columns are exactly I^c together with Γ . $\square$

Corollary 11.5 (Complementary-minor criterion). *For fixed E_0, O_0 , some orientation has nonzero determinant if and only if there exists $I \subseteq [r]$, $|I| = |O_0|$, such that*

$$\det V[O_0, I] \neq 0, \quad \det U[E_0, I^c \cup \Gamma] \neq 0.$$

Proof. The monomials $\prod_{i \in I} s_i$ are distinct characters of the Boolean cube and hence linearly independent as functions on $\{\pm 1\}^r$. Thus (15) is not identically zero exactly when one coefficient is nonzero. $\square$

This is the sharpest reduction in the present paper. On the pair-index set $[r]$, the odd matrix $V[O_0, *]$ defines a representable matroid. After contracting the fixed self-conjugate columns Γ , the even matrix defines another. The desired I must be independent of full size on the odd side while its complement completes Γ to an even basis. The informal requirement that capacity “cannot be reused” is therefore exactly encoded by complementarity.

Problem 11.6 (Complementary-Minor Capacity Problem). For every relevant λ , prove that the favorable support S_λ contains $E_0 \subseteq E_n$ and $O_0 \subseteq O_n$ with $|E_0| + |O_0| = h_n$, together with $I \subseteq [r_n]$, $|I| = |O_0|$, for which both complementary minors in Theorem 11.5 are nonzero.

A solution of Theorem 11.6 proves OTT, hence (6), hence OPAC-038 in that degree. The natural next step is a Hall–Rado or matroid-intersection rank criterion derived from character/fusion structure, not generic matroid theory alone.

Adaptive orientation is essential.

Proposition 11.7 (No universal fixed orientation). *At $n = 4$ and $\lambda = (2, 1, 1)$, the fixed selected columns $(4), (3, 1), (2, 2)$ fail the oriented-transversal test, while replacing $(3, 1)$ by its conjugate $(2, 1, 1)$ restores full rank.*

Proof. For $\lambda' = (3, 1)$ the favorable rows are $(3, 1), (2, 1, 1), (1^4)$. The first selection gives

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 3 & 2 \end{pmatrix},$$

which has rank 2. The reoriented selection has rank 3. $\square$

12. LONG-CYCLE GEOMETRY AND A SELECTED-RANK MODEL

For completeness we record two additional exact frameworks that constrain possible proofs. Suppose n is even. Full-cycle factorization theory gives a parity-reversing Ferrers graph: a class α can multiply with β to an n -cycle precisely when the parities are opposite and $\ell(\alpha) + \ell(\beta) \leq n + 1$; see Goupil–Schaeffer [5] and Poulalhon–Schaeffer [11].

Theorem 12.1 (Unsigned long-cycle matching). *For even n , the Ferrers graph on $B_n \sqcup O_n$ with edge condition*

$$\nu \sim \beta \iff \ell(\nu) + \ell(\beta) \leq n + 1$$

has a perfect matching.

Proof sketch. Shell the two sides by colength and length. Subtracting 1 from every part converts the shell counts into restricted partition counts. Elementary injections show the required initial-sum inequalities, which are Hall’s conditions for the Ferrers graph. $\square$

On the boundary $\ell(\alpha) + \ell(\beta) = n + 1$, the genus-zero full-cycle factorization formula gives a rank-one block. This ambient matching geometry does not survive arbitrary sign selection, so it is not itself a proof of parity balance.

A second model uses the critical Jucys kernel. At $M = n - 1$, the content product $P_\rho(M)$ vanishes only for the sign representation. If X is the character table and $D = \text{diag}(P_\rho(n - 1))$, set $C = X^T D X$. A sufficient selected-rank condition is

$$(16) \quad \text{rank } C[L_\lambda, E_\lambda] = |L_\lambda|.$$

This condition is stronger than parity balance and remains open.

The reciprocal-content kernel is a class-coordinate Moore–Penrose inverse. Standard Weingarten/Jucys machinery appears in Collins–Śniady [3], Novak [9], and Zinn–Justin [15]. With $Z = \text{diag}(z_\mu)$, $\varepsilon = (\varepsilon(\mu))$, $v = Z^{-1}\varepsilon$, and reciprocal diagonal D^+ , define

$$R = Z^{-1} X^T D^+ X Z^{-1}.$$

Then

$$\begin{aligned} \ker C &= \langle v \rangle, & \ker R &= \langle \varepsilon \rangle, \\ CR &= I - \varepsilon v^T, & RC &= I - v \varepsilon^T. \end{aligned}$$

Bordering by the null vectors and applying the complementary-submatrix nullity theorem [12] converts failure of (16) into an explicit dual flatness obstruction. This is useful diagnostically, but the selected-rank conjecture is not used as a theorem here.

13. NO-GO RESULTS

Several plausible strengthenings are false. Recording them prevents circular or insufficient arguments.

- N1. Componentwise half-negativity fails.** At $n = 5$, $\lambda = (2, 2, 1)$, a two-vertex Glaisher component can have values $(-1, -1)$. Deficit payment must therefore be nonlocal across components.
- N2. Commuting Hall matching is too restrictive.** Although cyclic extensions admit class-capacity matchings of the type studied by Britnell–Wildon [1, 2], a negative even class need not have a commuting odd class of favorable sign.
- N3. One-fibre coarsening positivity fails.** The Fourier coefficients in (13) can be negative, and an entire binary coarsening fibre can be unfavorable.
- N4. Scalar moments do not determine sign counts.** Even all inequalities in Theorem 9.1 admit abstract countermodels with too many negatives on one side.
- N5. Generic split-fusion matrices do not imply the required inertia bound.** Entrywise nonnegativity plus a diagonal difference is insufficient without fusion-ring structure.
- N6. Antisymmetric-only Kronecker certificates fail.** For a conjugate pair ν, ν' ,

$$(e_\nu - e_{\nu'})^T K_\lambda (e_\nu - e_{\nu'}) = 2(g(\nu, \nu, \lambda) - g(\nu, \nu, \lambda')),$$

so positivity of $g(\nu, \nu, \lambda)$ alone does not control the quadratic form.

- N7. Nested spectral selectors are tautological without module coupling.** Simultaneous character-table diagonalization merely selects already-known eigenspaces.
- N8. A single local split rule is impossible.** Theorem 6.4 gives an infinite family defeating every canonical two-part odd refinement.
- N9. A fixed oriented transversal is impossible.** Theorem 11.7 shows that the orientation must adapt to the row.

These failures point to the same requirement: the missing theorem must be unweighted, nonlocal, constituent-sensitive, and capacity-aware.

14. LOGICAL STATUS AND REMAINING FRONTIER

We summarize what is proved and what is not.

14.1. Proved global statements. The parity reduction Theorem 2.2, asymptotic self-conjugate elimination Theorem 3.3, exact positive filter Theorems 4.1 and 4.2, positive central forcing Theorem 5.1, square-root parity localization Theorem 5.2, unweighted moment domination Theorem 9.1, constituent localizers Theorem 9.2, split-fusion identity Theorem 7.2, three-sector decomposition Theorem 10.1, restricted-table reduction Theorem 11.3, and determinant expansion Theorem 11.4 are exact symbolic statements.

14.2. Proved balanced-hook statements. The Glaisher-edge formula, injective positivity above one third, first-band classification, parity-complete absorption, infinite split-dead family, and binary-square descent are exact statements for the indicated hook family. They demonstrate both target abundance and collision obstructions but do not establish a global Hall matching.

14.3. Open statements. OPAC-038 itself remains open. The Oriented Transversal Theorem Theorem 11.2, the Complementary-Minor Capacity Problem Theorem 11.6, the selected-rank condition (16), and the balanced-hook global matching problem are not proved.

The complementary-minor problem is presently the narrowest sufficient target. It packages the earlier three-sector capacity problem into ordinary character-table linear algebra: the odd block consumes a set I of non-self-conjugate pair directions, the fixed self-conjugate columns are forced onto the even side, and the complement I^c prevents reuse of capacity. The split-fusion surplus and constituent-level balanced localizers are therefore best viewed as tools for proving the subset-rank inequalities required by this complementary-basis problem.

15. CONCLUSION

The unweighted nature of OPAC-038 makes standard orthogonality and spectral methods insufficient by themselves. The results here progressively remove the mismatches. Parity identifies the exact capacity deficit. Self-conjugate rows are asymptotically harmless. The rising-factorial filter converts signs into support of a genuine positive-semidefinite fusion matrix. Binary Glaisher compression supplies a tractable local model with exact injections, absorption reservoirs, and infinite split obstructions. Alternating-group restriction turns each exceptional distinct-odd value into an exact fusion-channel imbalance. Schur-positive class sums cross the weighted-to-unweighted barrier and retain constituent information through nonnegative multiplication matrices.

The final reduction is combinatorial-linear rather than merely spectral. Adaptive conjugate-pair orientations turn the desired rank into a restricted character-table transversal, and Laplace expansion shows that the certificate exists exactly when complementary odd and even minors can be made simultaneously nonzero. This is a representable matroid-capacity problem with the no-reuse condition built in. Establishing the corresponding subset-rank inequalities from symmetric-group fusion structure would prove the desired extremal bound. At present that coupling remains open.

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REFERENCES

- [1] J. R. Britnell and M. Wildon, *On the distribution of conjugacy classes between the cosets of a finite group in a cyclic extension*, Bull. Lond. Math. Soc. **40** (2008), 897–906. doi:10.1112/blms/bdn073.
- [2] J. R. Britnell and M. Wildon, *Commuting conjugacy classes: an application of Hall’s marriage theorem to group theory*, J. Group Theory **12** (2009), 795–802. doi:10.1515/JGT.2009.013.
- [3] B. Collins and P. Śniady, *Integration with respect to the Haar measure on unitary, orthogonal and symplectic groups*, Comm. Math. Phys. **264** (2006), 773–795. doi:10.1007/s00220-006-1554-3.
- [4] B. Fristedt, *The structure of random partitions of large integers*, Trans. Amer. Math. Soc. **337** (1993), 703–735. doi:10.1090/S0002-9947-1993-1094553-1.
- [5] A. Goupil and G. Schaeffer, *Factoring n -cycles and counting maps of given genus*, European J. Combin. **19** (1998), 819–834. doi:10.1006/eujc.1998.0215.
- [6] G. H. Hardy and S. Ramanujan, *Asymptotic formulae in combinatory analysis*, Proc. London Math. Soc. (2) **17** (1918), 75–115. doi:10.1112/plms/s2-17.1.75.
- [7] A. A. Jucys, *Symmetric polynomials and the center of the symmetric group ring*, Rep. Math. Phys. **5** (1974), 107–112. doi:10.1016/0034-4877(74)90019-6.
- [8] K. Kawsathon and K. Rodtes, *On the character tables of symmetric groups*, Algebra Discrete Math. **32** (2021), 211–225. doi:10.12958/adm1490.
- [9] J. I. Novak, *Jucys–Murphy elements and the unitary Weingarten function*, Banach Center Publ. **89** (2010), 231–235. doi:10.4064/bc89-0-14.
- [10] I. Pak, G. Panova and E. Vallejo, *Kronecker products, characters, partitions, and the tensor square conjectures*, Adv. Math. **288** (2016), 702–731. doi:10.1016/j.aim.2015.11.002.

- [11] D. Poulalhon and G. Schaeffer, *Factorizations of large cycles in the symmetric group*, Discrete Math. **254** (2002), 433–458. doi:10.1016/S0012-365X(01)00361-2.
- [12] G. Strang and T. Nguyen, *The interplay of ranks of submatrices*, SIAM Rev. **46** (2004), 637–646. doi:10.1137/S0036144503434381.
- [13] S. Sundaram, *The conjugacy action of S_n and modules induced from centralisers*, J. Algebraic Combin. **48** (2018), 179–225. doi:10.1007/s10801-017-0796-9.
- [14] M. Wildon, *A combinatorial proof of a plethystic Murnaghan–Nakayama rule*, SIAM J. Discrete Math. **30** (2016), 1526–1533. doi:10.1137/14098260X.
- [15] P. Zinn–Justin, *Jucys–Murphy elements and Weingarten matrices*, Lett. Math. Phys. **91** (2010), 119–127. doi:10.1007/s11005-009-0365-9.

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